COLLÈGE
DE FRANCE
1530

Chaire de Physique Mésoscopique
Michel Devoret
Année 2010, 11 mai - 22 juin

INTRODUCTION AU CALCUL QUANTIQUE

INTRODUCTION TO QUANTUM COMPUTATION

Quatrième Leçon / *Fourth Lecture*

VISIT THE WEBSITE OF THE CHAIR OF MESOSCOPIC PHYSICS

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http://www.physinfo.fr/lectures.html

PDF FILES OF ALL PAST LECTURES ARE POSTED

Questions, comments and corrections are welcome!

write to "phymeso@gmail.com"

CONTENT OF THIS YEAR'S LECTURES

QUANTUM COMPUTATION FROM THE PERSPECTIVE OF MESOSCOPIC CIRCUITS

1. Introduction, c-bits versus q-bits
2. The Pauli matrices and quantum computation primitives
3. Stabilizer formalism for state representation
4. Clifford calculus
5. Algorithms
6. Error correction

CALENDAR OF SEMINARS

May 11: Cristian Urbina, (Quantronics group, SPEC-CEA Saclay)
Josephson effect in atomic contacts and carbon nanotubes

May 18: Benoît Douçot (LPTHE / Université Pierre et Marie Curie)
Towards the physical realization of topologically protected qubits

June 1: Takis Kontos (LPA / Ecole Normale Supérieure)
Points quantiques et ferromagnétisme

June 8: Cristiano Ciuti (MPQ, Université Paris - Diderot)
Ultrastrong coupling circuit QED : vacuum degeneracy and quantum phase transitions

June 15: Leo DiCarlo (Yale)
Preparation and measurement of tri-partite entanglement in a superconducting quantum circuit

June 22: Vladimir Manucharian (Yale)
The fluxonium circuit: an electrical dual of the Cooper-pair box?

LECTURE IV : CLIFFORD CALCULUS

1. Review of stabilizer properties
2. Number of stabilizers
3. One-qubit Clifford operations
4. Two-qubit interactions
5. Two-qubit Clifford operations

OUTLINE

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BASIC INGREDIENTS OF STABILIZER FORMALISM

Primary Pauli operators: $P_1 P_2 .... P_N$ $\qquad P_i \in \{I, Z, X, Y\}$

N : number of qubits

N =6 example: $ZIZIXX$

Stabilizer element: $\pm P_1 P_2 .... P_N = M \neq -I$

Stabilizer: $\{M_1, M_2, ..., M_{2^N-1}\}$ $\qquad \forall \{j,k\}, M_j M_k M_j M_k = I$

represents a state of the register

Condensed form: $\underset{\longleftarrow\ N \text{ elements}\ \longrightarrow}{\{M_\alpha, M_\beta, ..., M_\nu\}}$ $\forall \{\alpha, \beta, \gamma\}, M_\alpha M_\beta \neq \pm M_\gamma$

A D-dimensional state manifold of the register is represented by:

$\underset{\longleftarrow\ P \text{ elements}\ \longrightarrow}{\{M_\alpha, M_\beta, ..., M_\mu\}}$ $\qquad D = N - P$

STABILIZER ↔ REGISTER STATE

A single-state stabilizer can be written as an array of symbols:

$$\left.\boxed{\begin{array}{ccccc} \pm & P_{11} & P_{12} & \cdots & P_{1N} \\ \pm & P_{21} & P_{22} & \cdots & P_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \pm & P_{N1} & P_{N2} & \cdots & P_{NN} \end{array}}\right\}$$

N Pauli words
with N+1 symbols
arranged in "alphabetical"
order
(I-Z-X-Y)

The associated state simultaneously diagonalizes the operator-words with + 1 eigenvalue:

$$\boxed{\begin{array}{c} +IZ \\ +ZI \end{array}} \; |00\rangle \qquad \boxed{\begin{array}{c} +IX \\ +XI \end{array}} \; \frac{(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)}{2} = |++\rangle \qquad \boxed{\begin{array}{c} -ZZ \\ -XX \end{array}} \; \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

$$\boxed{\begin{array}{c} +IZ \\ -ZI \end{array}} \; |10\rangle \qquad \boxed{\begin{array}{c} -IX \\ +XI \end{array}} \; \frac{(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)}{2} = |+-\rangle \qquad \boxed{\begin{array}{c} -ZZ \\ +XX \end{array}} \; \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

STABILIZER – STATE MAPPING (continued)

$$\boxed{\begin{array}{l}+ZZ\\+XX\end{array}} \quad \frac{|00\rangle+|11\rangle}{\sqrt{2}}=\frac{(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)+(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)}{2\sqrt{2}}=\frac{|++\rangle-|--\rangle}{\sqrt{2}}$$

remove signs $\longrightarrow \boxed{\begin{array}{l}ZZ\\XX\end{array}} \longrightarrow \left\{\frac{|00\rangle+|11\rangle}{\sqrt{2}},\frac{|00\rangle-|11\rangle}{\sqrt{2}},\frac{|01\rangle+|10\rangle}{\sqrt{2}},\frac{|01\rangle-|10\rangle}{\sqrt{2}}\right\}$

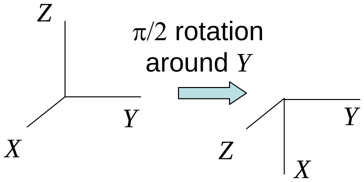
less is more !

$\boxed{\begin{array}{l}+ZX\\-XZ\end{array}} \longrightarrow$? Observe that: $\boxed{\begin{array}{l}+ZZ\\+XX\end{array}} \xrightarrow{R_{y2}\left(\frac{\pi}{2}\right)} \boxed{\begin{array}{l}+ZX\\-XZ\end{array}}$

$$\downarrow$$

$$\frac{|00\rangle+|01\rangle-|10\rangle+|11\rangle}{2}$$

since



STABILIZER – STATE MAPPING (continued)

$\boxed{\begin{array}{l}+IZ\\+ZI\end{array}}$ $|00\rangle$

$+ZZ$ ← pivot

$\boxed{\begin{array}{l}+ZZ\\+XX\end{array}}$ $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$

$\boxed{\begin{array}{l}+ZZ\\-XX\end{array}}$ $\frac{|00\rangle-|11\rangle}{\sqrt{2}}$

$$XX\,|00\rangle=|11\rangle$$

$$XX\,|11\rangle=|00\rangle$$

$\boxed{\begin{array}{l}+IIZ\\+IZI\\+ZII\end{array}}$ $|000\rangle$

$+IZZ$ ← pivot

$+ZIZ$ ← pivot

$+ZZI$

$+ZZZ$

$\boxed{\begin{array}{l}+IZZ\\+ZIZ\\+XXX\end{array}}$ $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$

$\boxed{\begin{array}{l}+IZZ\\+ZIZ\\-XXX\end{array}}$ $\frac{|000\rangle-|111\rangle}{\sqrt{2}}$

$$XXX\,|000\rangle=|111\rangle$$

$$XXX\,|111\rangle=|000\rangle$$

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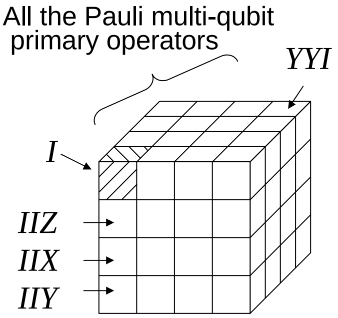
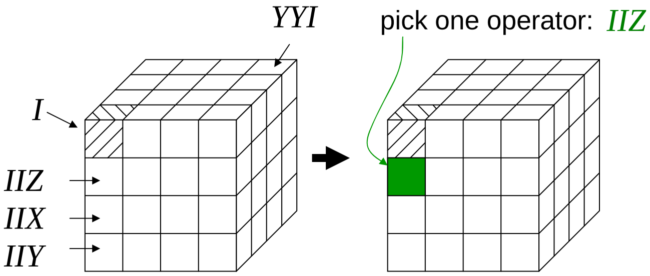
CONSTRUCTION OF A STABILIZER CLASS

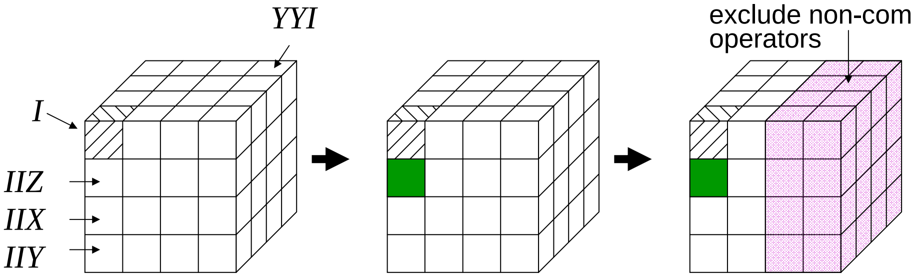


CONSTRUCTION OF A STABILIZER CLASS



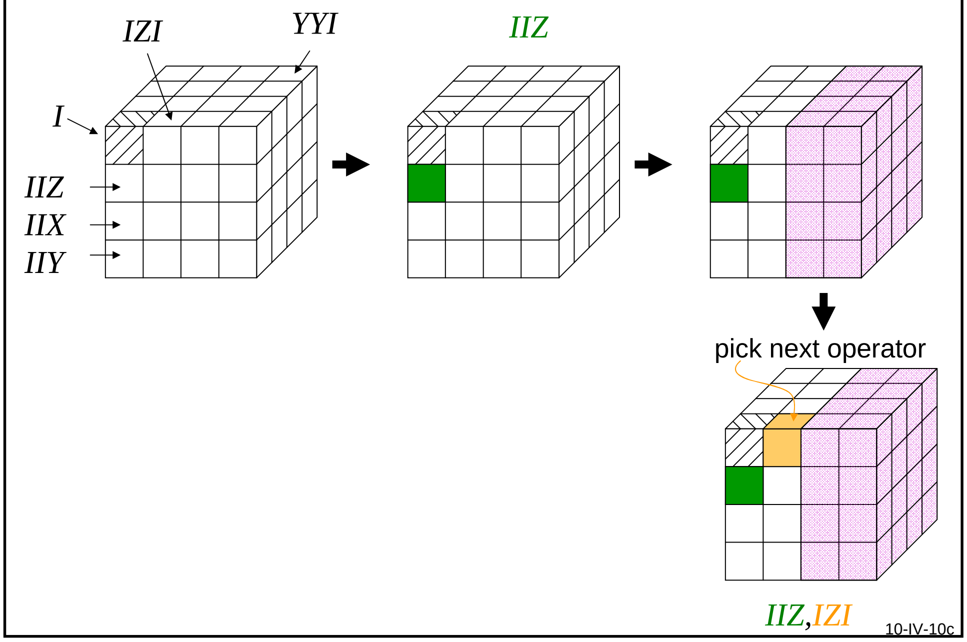
pick one operator: *IIZ*

CONSTRUCTION OF A STABILIZER CLASS



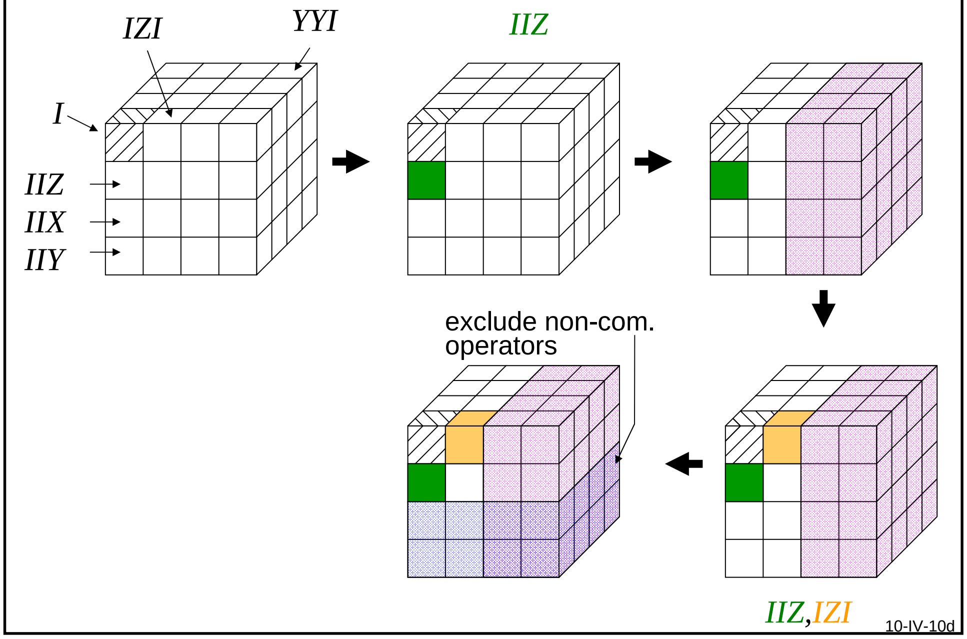
exclude non-com.
operators

CONSTRUCTION OF A STABILIZER CLASS



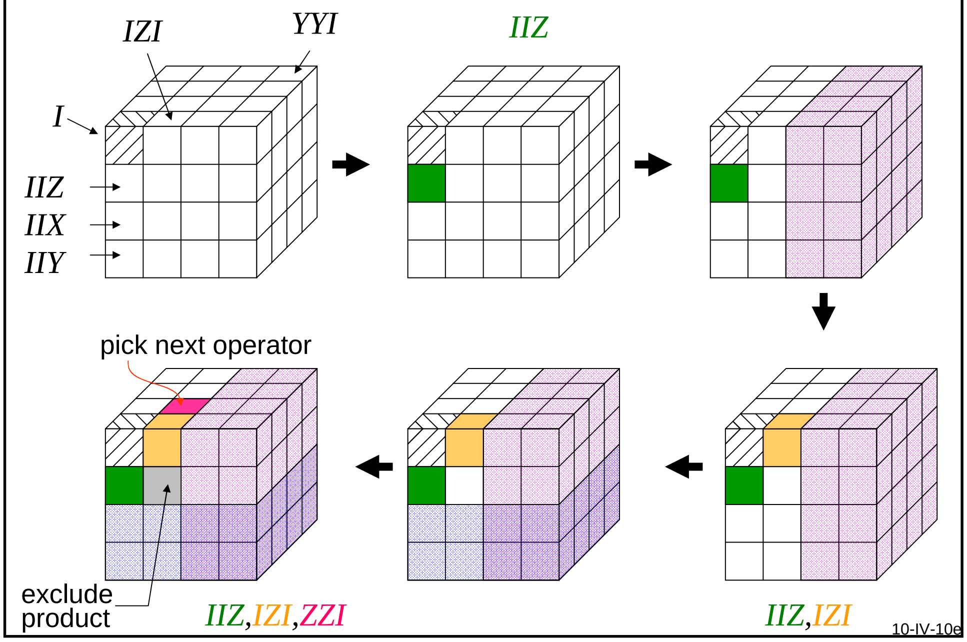
pick next operator

CONSTRUCTION OF A STABILIZER CLASS

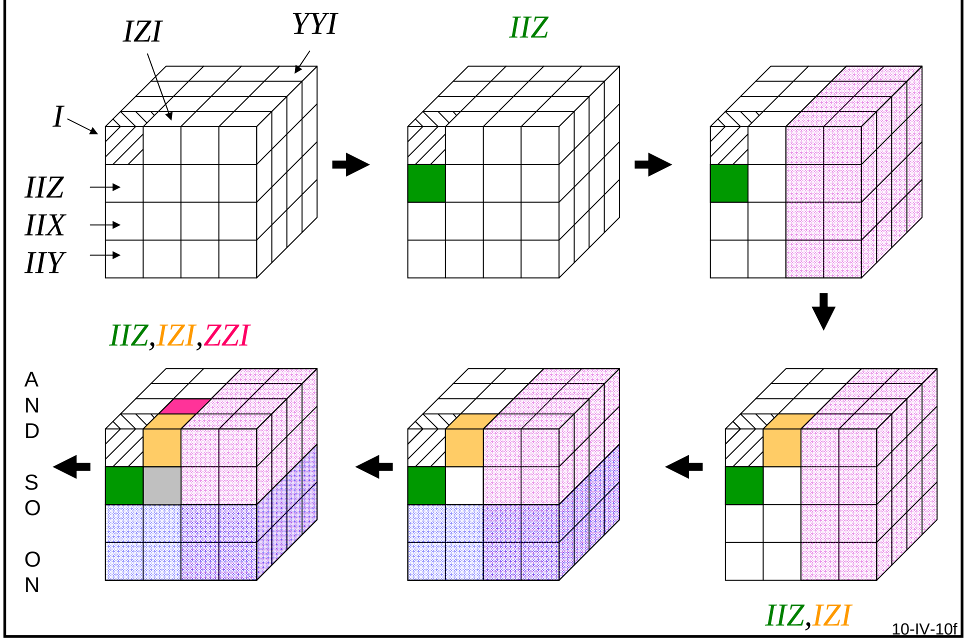


exclude non-com. operators

CONSTRUCTION OF A STABILIZER CLASS



CONSTRUCTION OF A STABILIZER CLASS



NUMBER OF STABILIZER CLASSES

First choice: $2^{2N}-1$ the -1 comes from excluding I

Second choice: $2^{2N-1}-2$ exclude non.com., last gen. and I

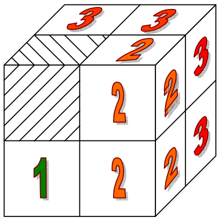
Third choice: $2^{2N-2}-4$ exclude non.com., last gen. comb. and I

$\vdots$

k-th choice: $2^{2N-k+1}-2^{k-1}$

Total nb. choices: $\prod_{k=1}^{N}\left(2^{2N-k+1}-2^{k-1}\right)=\prod_{k=1}^{N}\left(2^{N-k+1}+1\right)\left(2^{N}-2^{k-1}\right)$

Last factor corresponds to the number of arrangements of numbers 1 thru N on hyper-cube of stabilizer gen. combinations. It must be divided out to get the classes.



The number of stabilizer classes is thus:

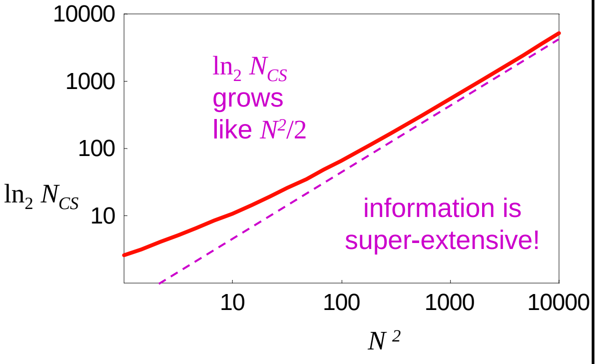
$$\prod_{k=1}^{N}\left(2^{N-k+1}+1\right)=\left(2^{N}+1\right)\left(2^{N-1}+1\right).....\cdot 5\cdot 3$$

NUMBER OF CLIFFORD STATES

$$N_{CS}=2^{N}\left(2^{N}+1\right)\left(2^{N-1}+1\right).....\cdot 5\cdot 3$$

N	N_{CS}
1	6
2	60
3	1080
4	36720
5	2423520

Information in selecting one of these states: $\ln_2 N_{CS}$



Gottesman-Knill theorem

entanglement, Bell violations, teleportation, error correction, but...

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THE SINGLE QUBIT CLIFFORD GROUP

Consider isomorphisms of the Pauli group into itself

Each isomorphism is characterized by the set
of the images of the generators: $\{g_z, g_x\} \in \{Z, -Z, X, -X, Y, -Y\}^{\times 2}$

$$\begin{cases} Z \mapsto g_z \\ X \mapsto g_x \end{cases}$$

The two images must satisfy:

$$g_z g_x = -g_x g_z$$

There are therefore 6x4=24 isomorphisms, called elements of the 1-qubit Clifford group:

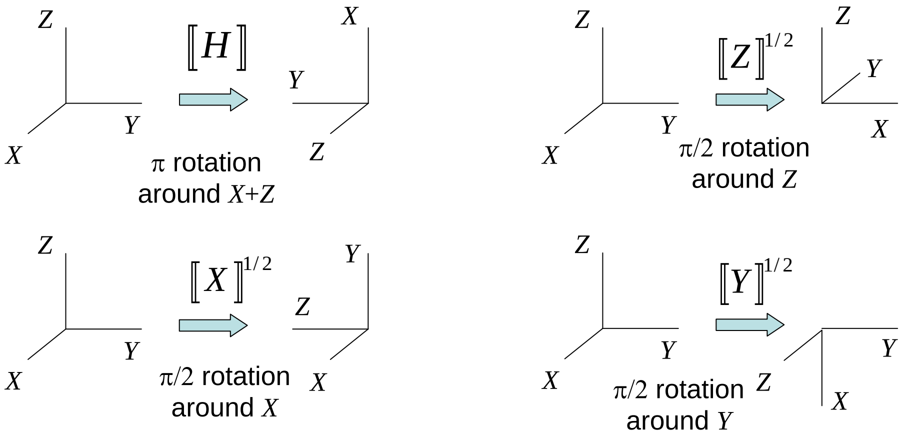
$$Id = \begin{Bmatrix} Z \mapsto Z \\ X \mapsto X \end{Bmatrix} \quad \llbracket Z \rrbracket^{1/2} = \begin{Bmatrix} Z \mapsto Z \\ X \mapsto Y \end{Bmatrix} \quad \llbracket X \rrbracket^{1/2} = \begin{Bmatrix} Z \mapsto -Y \\ X \mapsto X \end{Bmatrix} \quad \llbracket Y \rrbracket^{1/2} = \begin{Bmatrix} Z \mapsto X \\ X \mapsto -Z \end{Bmatrix}$$

$$\llbracket H \rrbracket = \begin{Bmatrix} Z \mapsto X \\ X \mapsto Z \end{Bmatrix} \quad \llbracket Z \rrbracket^{-1/2} = \begin{Bmatrix} Z \mapsto Z \\ X \mapsto -Y \end{Bmatrix} \quad \llbracket X \rrbracket^{-1/2} = \begin{Bmatrix} Z \mapsto Y \\ X \mapsto X \end{Bmatrix} \quad \llbracket Y \rrbracket^{-1/2} = \begin{Bmatrix} Z \mapsto -X \\ X \mapsto Z \end{Bmatrix}$$

$$\llbracket Z \rrbracket = \begin{Bmatrix} Z \mapsto Z \\ X \mapsto -X \end{Bmatrix} \quad \llbracket X \rrbracket = \begin{Bmatrix} Z \mapsto -Z \\ X \mapsto X \end{Bmatrix} \quad \llbracket Y \rrbracket = \begin{Bmatrix} Z \mapsto -Z \\ X \mapsto -X \end{Bmatrix}$$

Why $\llbracket P \rrbracket^{1/2}$ and not $[P]^{1/2}$? The latter is an operator acting on kets. The former is a super-operator acting on operators.

THE SINGLE QUBIT CLIFFORD GROUP IS ISOMORPHIC TO THE OCTAHEDRAL GROUP



π rotation around $X+Z$

$\pi/2$ rotation around Z

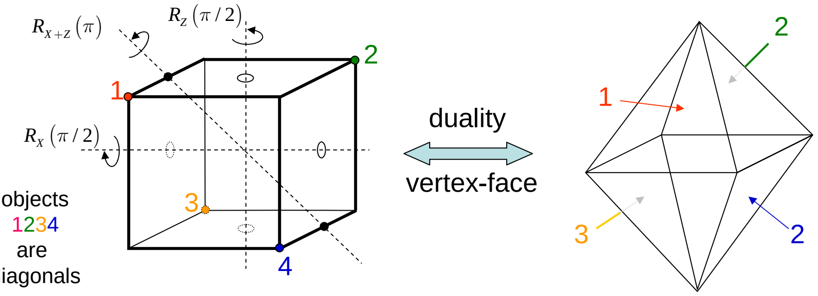
$\pi/2$ rotation around X

$\pi/2$ rotation around Y

The notation $[\![A]\!]^a$ means the transformation is a "rotation" around axis A with angle $a\pi$. Attention: $[\![A]\!]^2 = Id$

GENERATORS OF THE 1-QUBIT CLIFFORD GROUP (ISOMORPHIC TO THE OCTAHEDRAL GROUP)

Consider S_4 the permutation group on 4 objects, isomorphic to the octahedral group, symmetry group of the cube and the octahedron



objects 1234 are diagonals

objects 1234 are pairs of faces

1st choice of generators

$$R_Z(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = [\![Z]\!]^{1/2}$$

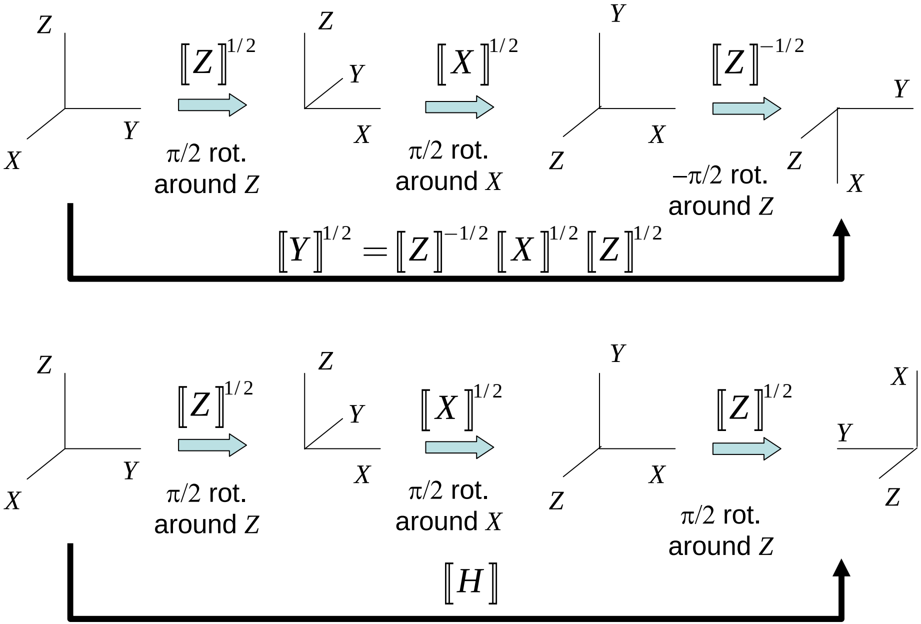
$$R_{X+Z}(\pi) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix} = [\![H]\!]$$

2nd choice of generators

$$R_Z(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = [\![Z]\!]^{1/2}$$

$$R_X(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = [\![X]\!]^{1/2}$$

USEFUL RELATIONS FOR QUANTUM COMPILERS



$$[\![Y]\!]^{1/2} = [\![Z]\!]^{-1/2}[\![X]\!]^{1/2}[\![Z]\!]^{1/2}$$

$$[\![H]\!]$$

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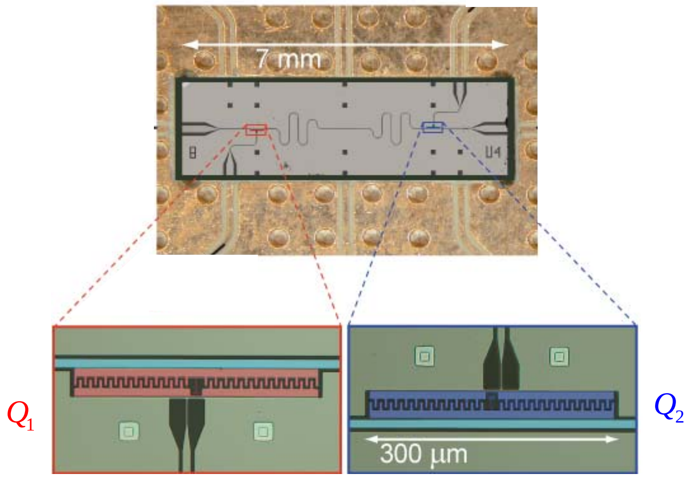
TWO-QUBIT QUANTUM PROCESSOR

slide courtesy of
Leo DiCarlo & Rob Schoelkopf
Yale University



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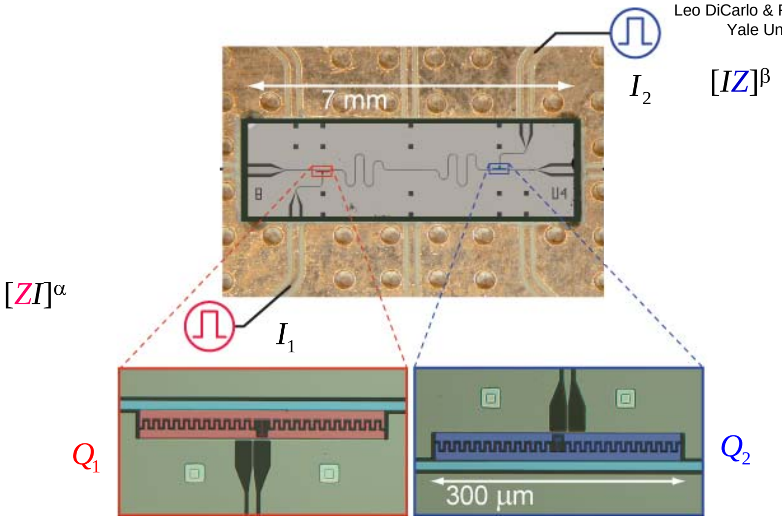

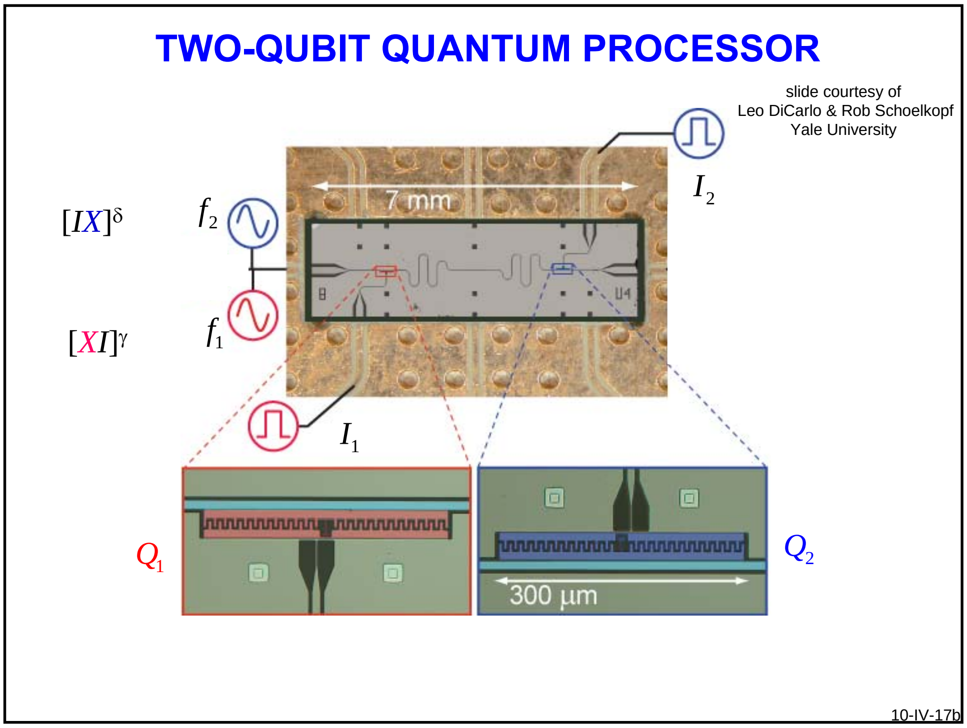

TWO-QUBIT QUANTUM PROCESSOR

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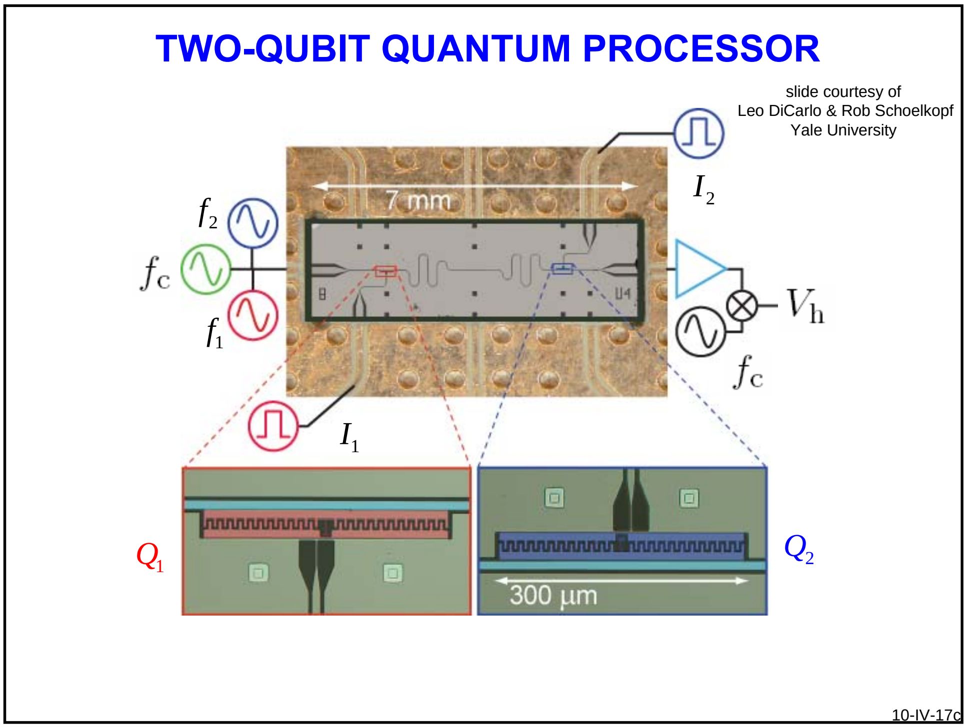
10-IV-17b



TWO-QUBIT QUANTUM PROCESSOR

slide courtesy of
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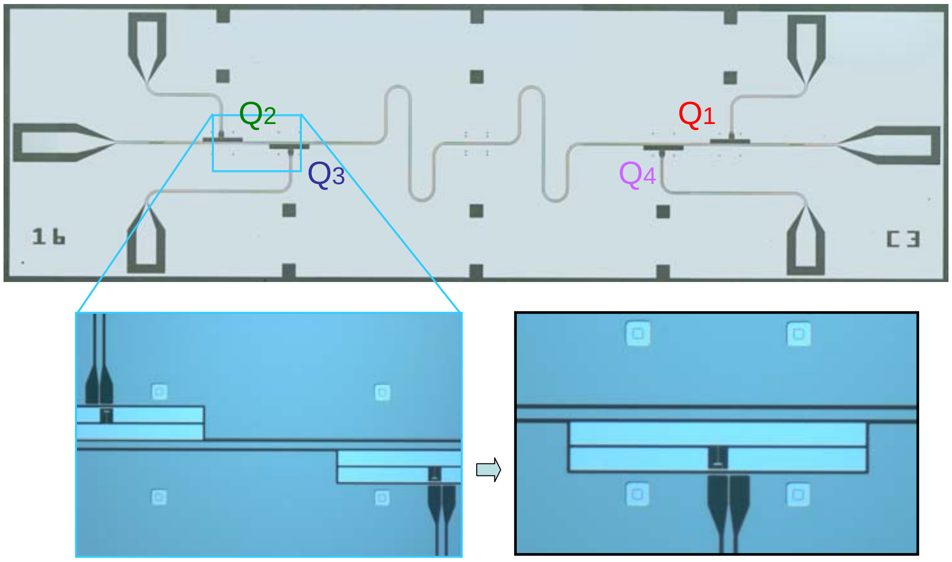
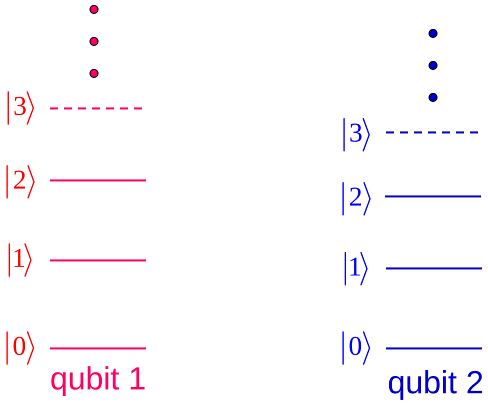
10-IV-17c

UPGRADING THE PROCESSOR TO 4 QUBITS

slide courtesy of Leo DiCarlo & Rob Schoelkopf

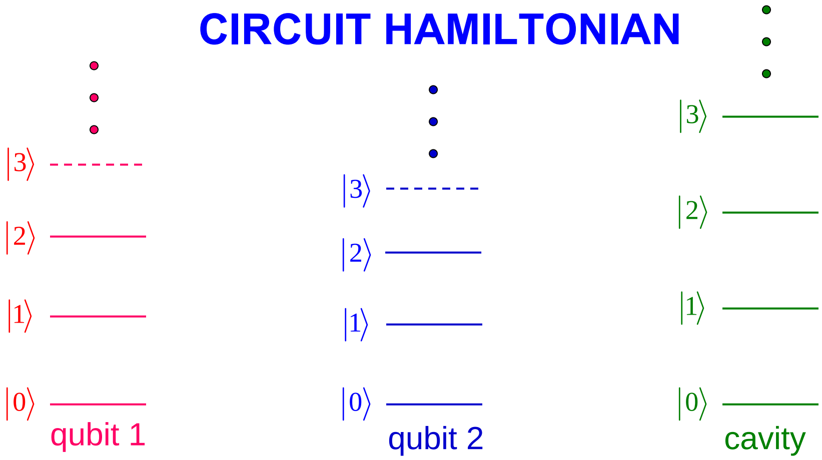


CIRCUIT HAMILTONIAN



$$\frac{H_1}{\hbar} = \omega_1 a_1^\dagger a_1 + \frac{\alpha_1}{2}\left(a_1^\dagger a_1\right)^2 \qquad \frac{H_2}{\hbar} = \omega_2 a_2^\dagger a_2 + \frac{\alpha_2}{2}\left(a_2^\dagger a_2\right)^2$$

CIRCUIT HAMILTONIAN



$$\frac{H_1}{\hbar} = \omega_1 a_1^\dagger a_1 + \frac{\alpha_1}{2}\left(a_1^\dagger a_1\right)^2$$

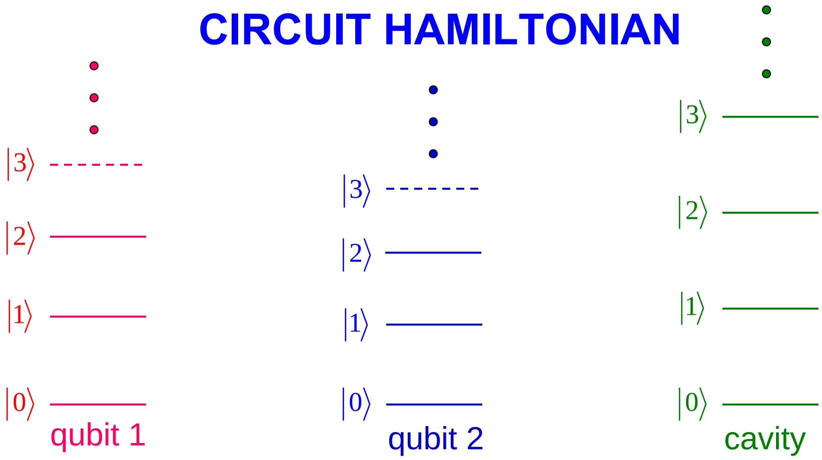
$$\frac{H_2}{\hbar} = \omega_2 a_2^\dagger a_2 + \frac{\alpha_2}{2}\left(a_2^\dagger a_2\right)^2$$

$$\frac{H_c}{\hbar} = \omega_c a_c^\dagger a_c$$

$$H_{\text{total}} = H_1 + H_2 + H_c + H_{\text{coupling}}$$

10-IV-19a

CIRCUIT HAMILTONIAN



$$\frac{H_1}{\hbar} = \omega_1 a_1^\dagger a_1 + \frac{\alpha_1}{2}\left(a_1^\dagger a_1\right)^2$$

$$\frac{H_2}{\hbar} = \omega_2 a_2^\dagger a_2 + \frac{\alpha_2}{2}\left(a_2^\dagger a_2\right)^2$$

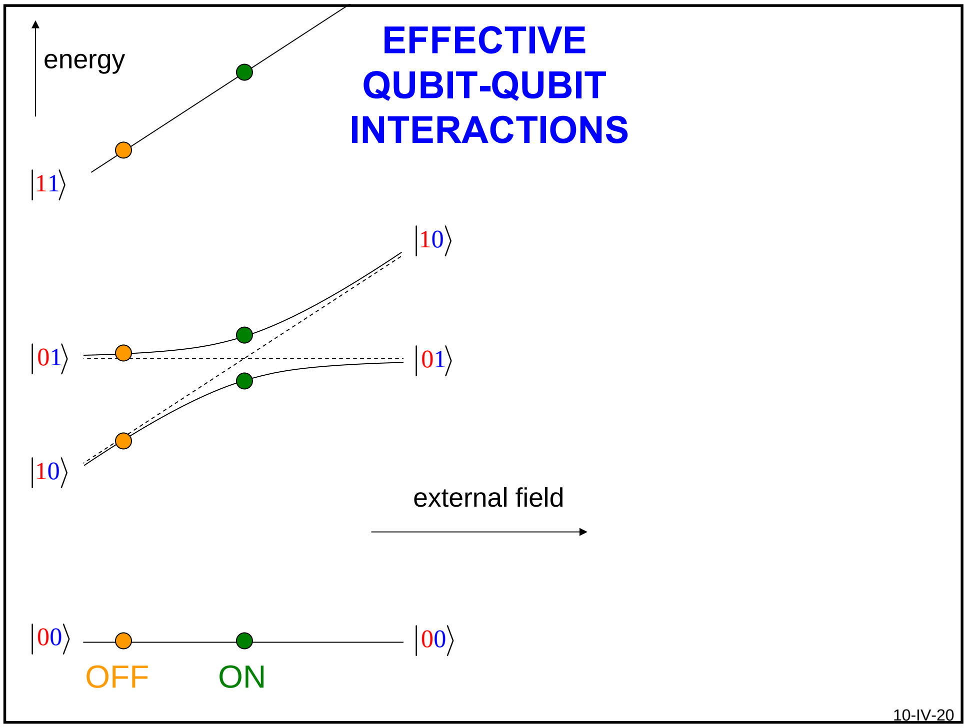
$$\frac{H_c}{\hbar} = \omega_c a_c^\dagger a_c$$

$$\frac{H_{\text{coupling}}}{\hbar} \simeq g_1\left(a_1^\dagger a_c + a_1 a_c^\dagger\right) + g_2\left(a_2^\dagger a_c + a_2 a_c^\dagger\right)$$

RWA

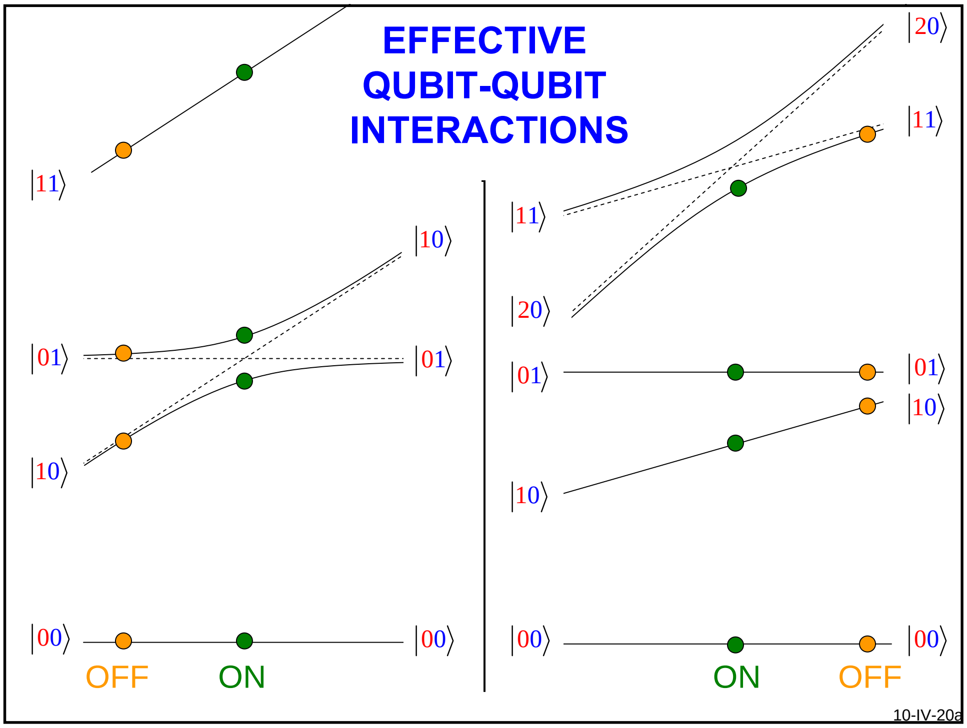
$$H_{\text{total}} = H_1 + H_2 + H_c + H_{\text{coupling}}$$

10-IV-19b



EFFECTIVE QUBIT-QUBIT INTERACTIONS

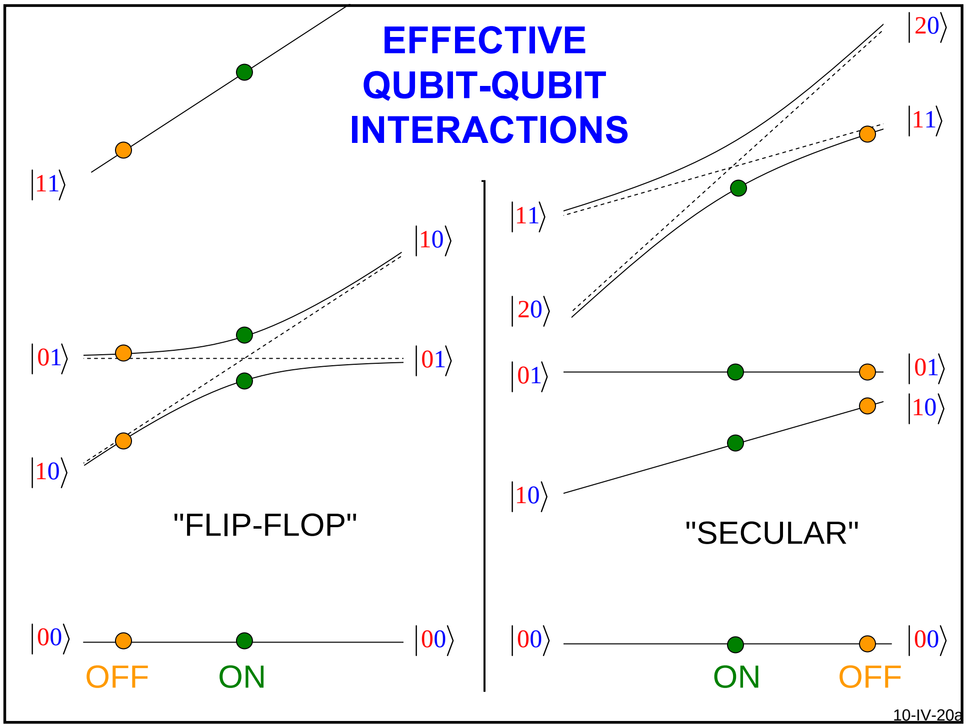
10-IV-20



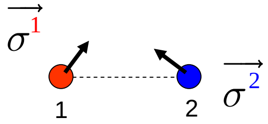
EFFECTIVE QUBIT-QUBIT INTERACTIONS

10-IV-20a

EFFECTIVE QUBIT-QUBIT INTERACTIONS



NATURAL ENTANGLING OPERATIONS



$$\hat{U}(\tau) = \exp\left(-i\hat{H}_{\text{int}}\tau/\hbar\right)$$

- * <u>Secular interaction</u>: $\hat{H}_{\text{int}} = g_{\parallel}\sigma_z^1\sigma_z^2 \longrightarrow [ZZ]^{1/2}$

with adjustment of gate duration time: $\tau_s = \dfrac{\pi\hbar}{4g_{\parallel}}$

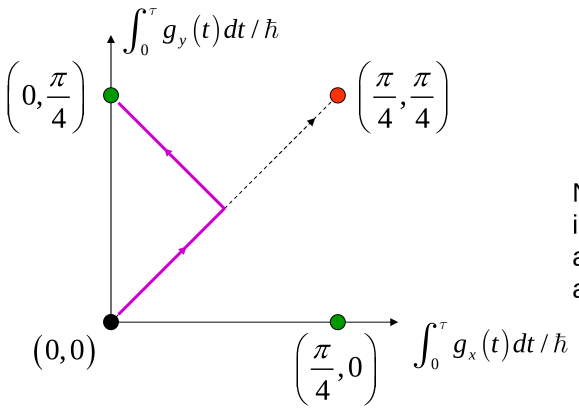
- * <u>Flip-flop interaction</u>: $\hat{H}_{\text{int}} = g_{\perp}\left(\sigma_+^1\sigma_-^2 + \sigma_-^1\sigma_+^2\right)$

$$= g_{\perp}\left(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2\right)/2$$

$$\tau_f = \frac{\pi\hbar}{4g_{\perp}} \longrightarrow [XX]^{1/4}[YY]^{1/4}$$

REFOCUS SEQUENCE NECESSARY WITH FLIP-FLOP INTERACTION

Consider $\hat{H}_{\text{int}} = g_x(t)\sigma_{1x}\sigma_{2x} + g_y(t)\sigma_{1y}\sigma_{2y}$



Green points:
$[ZZ]^{1/2}$ equivalent

Need to cut sequence in 2 sections, with a 180° flip of one qubit around x or y.

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GOTTESMAN TABLES FOR QUANTUM OPERATIONS

Bit Flip
(NOT)

π rotation
along $\hat{x}$

$[X]$	
Z	$-Z$
X	X

Phase Flip

π rotation
along $\hat{z}$

$[Z]$	
Z	Z
X	$-X$

Hadamard

π rotation
along $\hat{x}+\hat{z}$

H	
Z	X
X	Z

NOT$^{1/2}$

$\pi/2$ rotation
along $\hat{y}$

$[Y]^{1/2}$	
Z	X
X	$-Z$

Example of a 2-qubit gate:

$$|c,t\rangle \longrightarrow |c,t\oplus c\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$CNOT$	
IZ	ZZ
ZI	ZI
IX	IX
XI	XX

← phase kick-back

as in Boolean algebra

control → ← target

RULES OF CLIFFORD CALCULUS FOR N QUBITS

They are found starting from: $\llbracket B \rrbracket^{\alpha} A = [B]^{-\alpha} A [B]^{\alpha}$

$$\llbracket B \rrbracket^{1} [A] = [-A]$$ if A and B anticommute

$$= [A]$$ if A and B commute

$$\llbracket B \rrbracket^{1/2} [A] = [B][A]$$ if A and B anticommute

$$= [A]$$ if A and B commute

NOTE THAT: $\llbracket -B \rrbracket^{\alpha} = \llbracket B \rrbracket^{-\alpha}$

$$[A] = [B] \Leftrightarrow A = B$$

EXAMPLE OF CLIFFORD CALCULATIONS

$\llbracket IZ \rrbracket^{1/2}\ IX = IY$

$\llbracket IZ \rrbracket^{1/2}\ IY = -IX$

$\llbracket IY \rrbracket^{1/2}\ IZ = IX$

$\llbracket IY \rrbracket^{1/2}\ IX = -IZ$

$\llbracket ZI \rrbracket^{1/2}\ IX = IX$

$\llbracket YI \rrbracket^{1/2}\ IZ = IZ$

$\llbracket IZ \rrbracket^{1/2}\ XI = XI$

$\llbracket IY \rrbracket^{1/2}\ ZI = ZI$

$\llbracket ZI \rrbracket^{1/2}\ XI = YI$

$\llbracket ZI \rrbracket^{1/2}\ YI = -XI$

$\llbracket YI \rrbracket^{1/2}\ ZI = XI$

$\llbracket YI \rrbracket^{1/2}\ XI = -ZI$

$\llbracket ZZ \rrbracket^{1/2}\ IX = ZY$

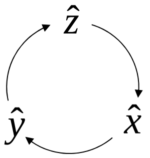
$\llbracket ZZ \rrbracket^{1/2}\ XI = YZ$

$\llbracket ZZ \rrbracket^{1/2}\ XX = XX$

$\llbracket ZZ \rrbracket^{1/2}\ YY = YY$

Only need
to know:

$\hat{z} \times \hat{x} = \hat{y}$



EXAMPLE OF POWER OF STABILIZER FORMALISM (1)

How do we go from the Computational basis to the Sign basis?

$$\{IZ, ZI, ZZ\} \longrightarrow \{IX, XI, XX\}$$

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \qquad \{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle\}$$

For both qubit, Z must be changed into X
This is performed by a 90° rotation around Y (easier than H).

$$\{IZ, ZI, ZZ\} \xrightarrow{\llbracket YI \rrbracket^{1/2} \llbracket IY \rrbracket^{1/2}} \{IX, XI, XX\}$$

$\llbracket IY \rrbracket^{1/2}\ IZ = IX$

$\llbracket IY \rrbracket^{1/2}\ ZI = ZI$

$\llbracket YI \rrbracket^{1/2}\ ZI = XI$

$\llbracket YI \rrbracket^{1/2}\ IX = IX$

EXAMPLE OF POWER OF STABILIZER FORMALISM (2)

How do we go from the i-Sign basis to the Phase basis?

$$\{IY, YI, YY\} \longrightarrow \{ZX, XZ, YY\}$$

$$\left\{\frac{(|0\rangle+i|1\rangle)(|0\rangle+i|1\rangle)}{2}, \frac{(|0\rangle+i|1\rangle)(|0\rangle-i|1\rangle)}{2}, \frac{(|0\rangle-i|1\rangle)(|0\rangle+i|1\rangle)}{2}, \frac{(|0\rangle-i|1\rangle)(|0\rangle-i|1\rangle)}{2}\right\}$$

$$\left\{\frac{|00\rangle+|01\rangle+|10\rangle-|11\rangle}{2}, \frac{|00\rangle+|01\rangle-|10\rangle+|11\rangle}{2}, \frac{|00\rangle-|01\rangle+|10\rangle+|11\rangle}{2}, \frac{-|00\rangle+|01\rangle+|10\rangle+|11\rangle}{2}\right\}$$

We need an entanglement operator: $\llbracket ZZ \rrbracket^{1/2}$

$$\llbracket ZZ \rrbracket^{1/2}\, IY = -ZX$$

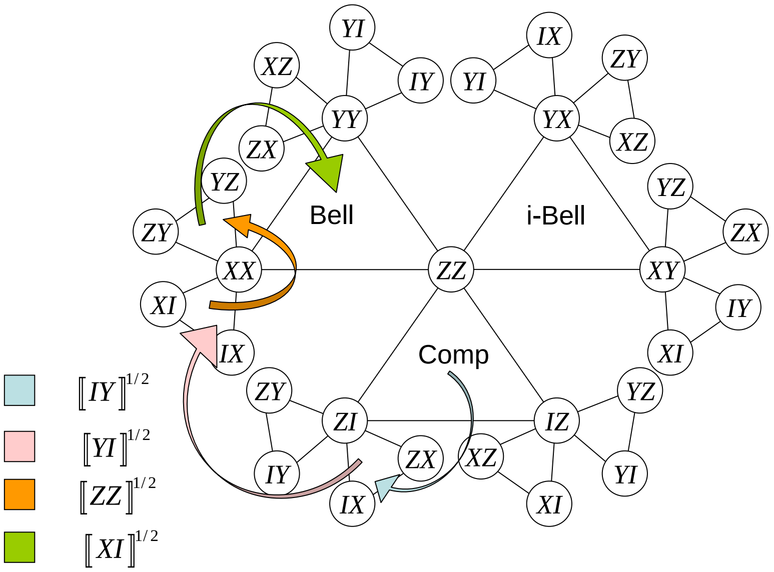
$$\llbracket ZZ \rrbracket^{1/2}\, YI = -XZ$$

$$\llbracket ZZ \rrbracket^{1/2}\, YY = YY$$

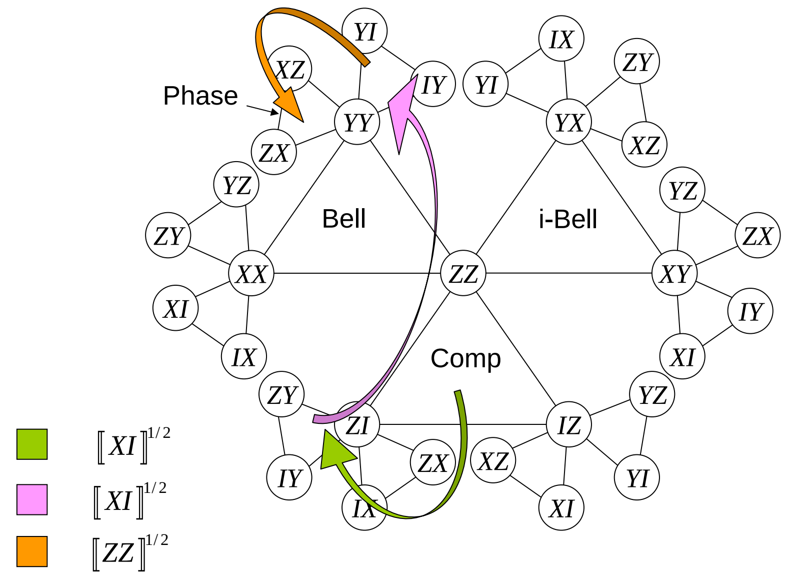
$$\{IY, YI, YY\} \xrightarrow{\llbracket ZZ \rrbracket^{1/2}} \{ZX, XZ, YY\}$$

signs unimportant in basis representation

VISUALIZING THE CLIFFORD MOVES ON THE STABILIZER MAP (1)



VISUALIZING THE CLIFFORD MOVES ON THE STABILIZER MAP (2)



🟩 $[\![XI]\!]^{1/2}$

🟪 $[\![XI]\!]^{1/2}$

🟧 $[\![ZZ]\!]^{1/2}$

10-IV-29

END OF LECTURE