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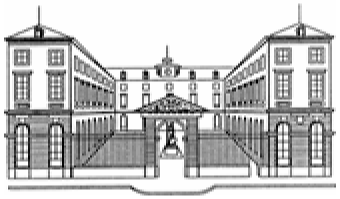


Chaire de Physique Mésoscopique
Michel Devoret
Année 2012, 15 mai - 19 juin

RÉSONATEURS NANOMÉCANIQUES DANS LE RÉGIME QUANTIQUE

NANOMECHANICAL RESONATORS IN QUANTUM REGIME

Deuxième leçon / *Second lecture*

Last update: 120525

PROGRAM OF THIS YEAR'S LECTURES

Lecture I: Introduction to nanomechanical systems

Lecture II: How do we model the coupling between electromagnetic modes and mechanical motion?

Lecture III: Is zero-point motion of the detector variables a limitation in the measurement of the position of a mechanical oscillator?

Lecture IV: In the active cooling of a nanomechanical resonator, is measurement-based feedback inferior to autonomous feedback?

Lecture V: How close to the ground state can we bring a nanoresonator?

Lecture VI: What oscillator characteristics must we choose to convert quantum information from the microwave domain to the optical domain?

CALENDAR OF 2012 SEMINARS

May 15: **Rob Schoelkopf (Yale University, USA)**
Quantum optics and quantum computation with superconducting circuits.

May 22: **Konrad Lehnert (JILA, Boulder, USA)**
Micro-electromechanics: a new quantum technology.

May 29: **Olivier Arcizet (Institut Néel, Grenoble)**
A single NV defect coupled to a nanomechanical oscillator: hybrid nanomechanics.

June 5: **Ivan Favero (MPQ, Université Paris Diderot)**
From micro to nano-optomechanical systems: photons interacting with mechanical resonators.

June 12: **A. Douglas Stone (Yale University, USA)**
Lasers and anti-lasers: a mesoscopic physicist's perspective on scattering from active and passive media.

June 19: **Tobias J. Kippenberg (EPFL, Lausanne, Suisse)**
Cavity optomechanics: exploring the coupling of light and micro- and nanomechanical oscillators.

LECTURE II : INTERACTION BETWEEN LIGHT AND MOVING MIRROR IN THE QUANTUM REGIME

OUTLINE

1. A simple hamiltonian, whose parameters span many orders of magnitude

2. Radiation pressure and the Doppler effect in a 1D deformable cavity: a fundamentally non-linear effect

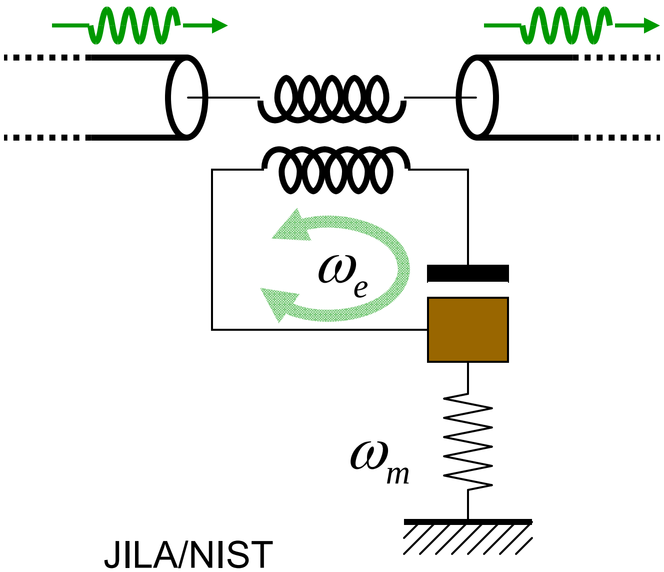
3. Full coupling hamiltonian between radiation and moving mirror

4. Adiabatic approximation

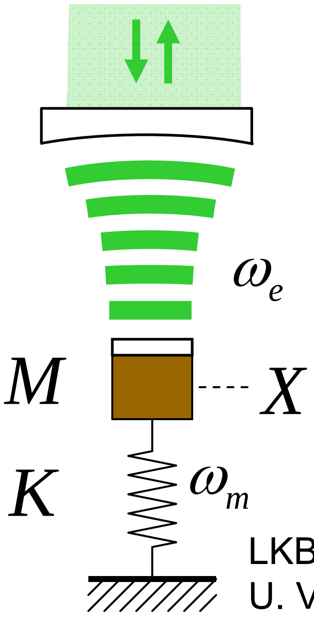
QUANTUM ELECTRO/OPTO-MECHANICS IS GOVERNED BY A SIMPLE HAMILTONIAN

$$\boxed{\frac{\hat{H}}{\hbar} = \omega_e \hat{a}^\dagger \hat{a} + \omega_m \hat{b}^\dagger \hat{b} + g_3 \left(\hat{b} + \hat{b}^\dagger\right) \hat{a}^\dagger \hat{a}}$$

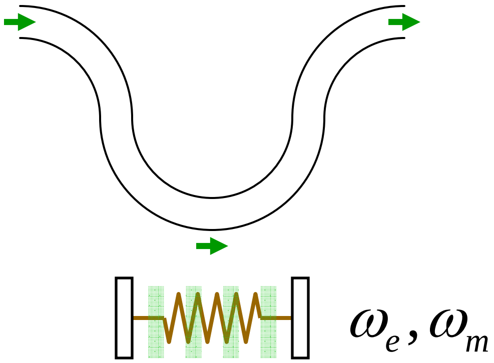
$$\omega_m = \sqrt{\frac{K}{M}} \quad Z_m = \sqrt{KM} \quad X_{ZPF} = \sqrt{\frac{\hbar}{2Z_m}} \quad \hat{X} = X_{ZPF}\left(b + b^\dagger\right) \quad g_3 = X_{ZPF}\frac{\omega_e}{\ell_0}$$



JILA/NIST

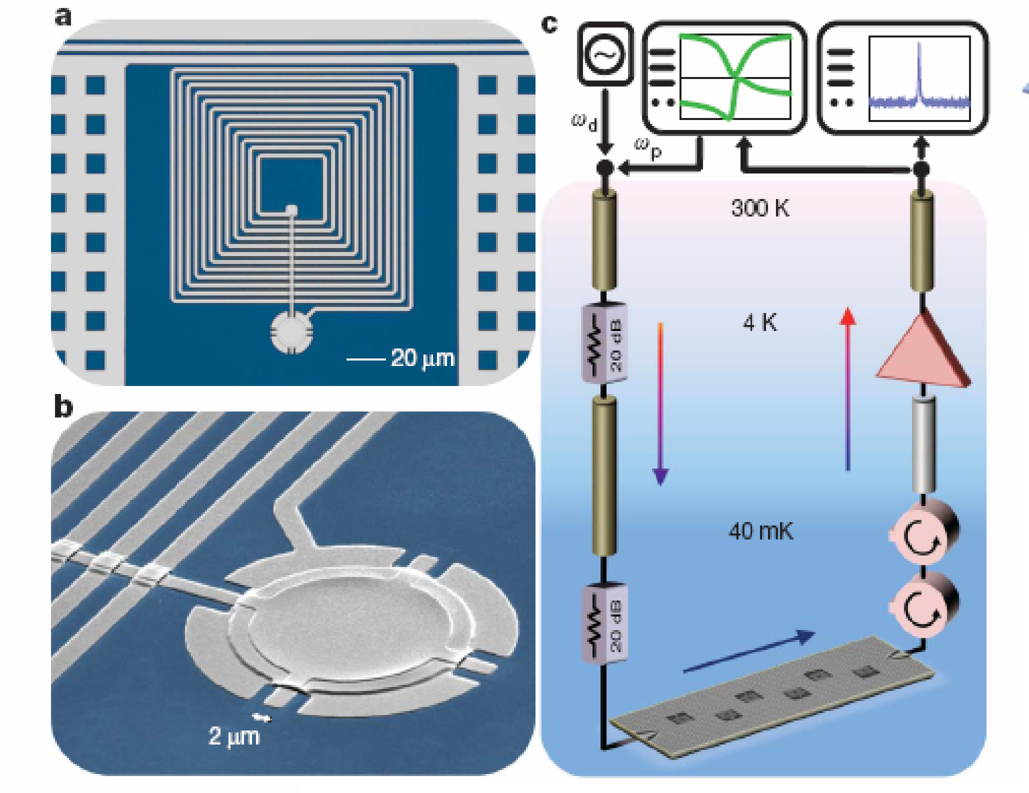


LKB-Paris,
U. Vienna



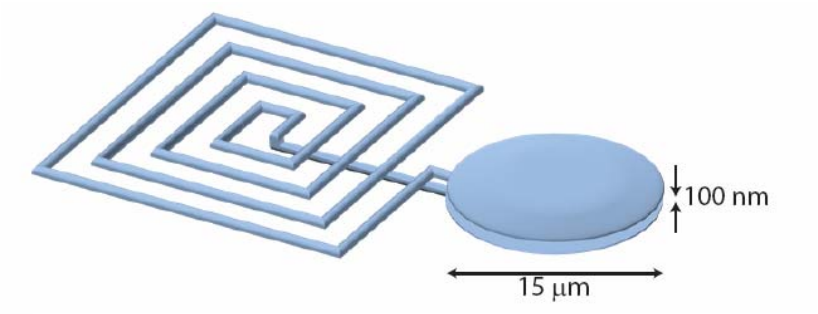
EPFL, Caltech
U. Vienna

QUANTUM ELECTROMECHANICS

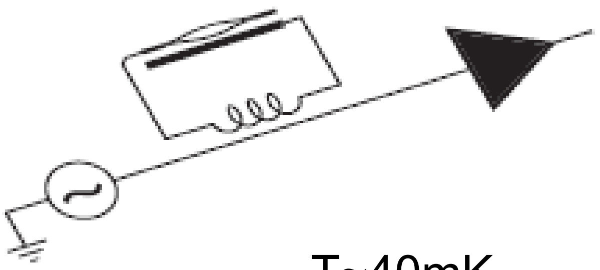


Teufel *et al*., Nature 471, 208 (2011)

JILA/NIST

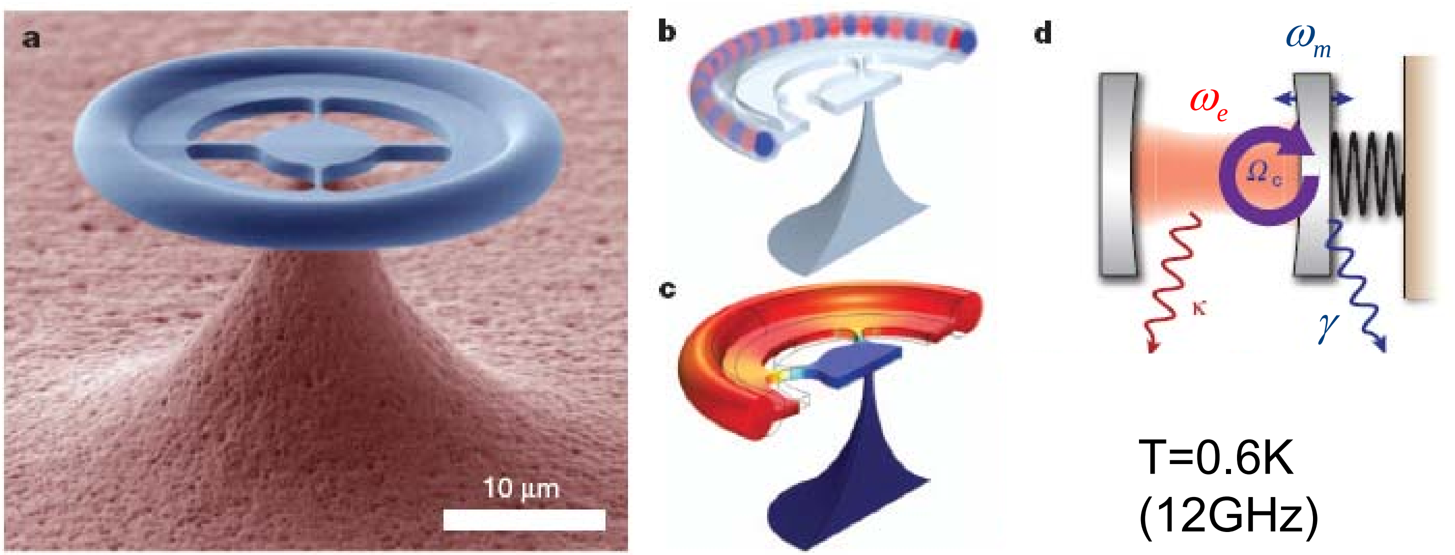


courtesy of K. Lenhert



T~40mK
(800MHz)

QUANTUM OPTOMECHANICS (1)



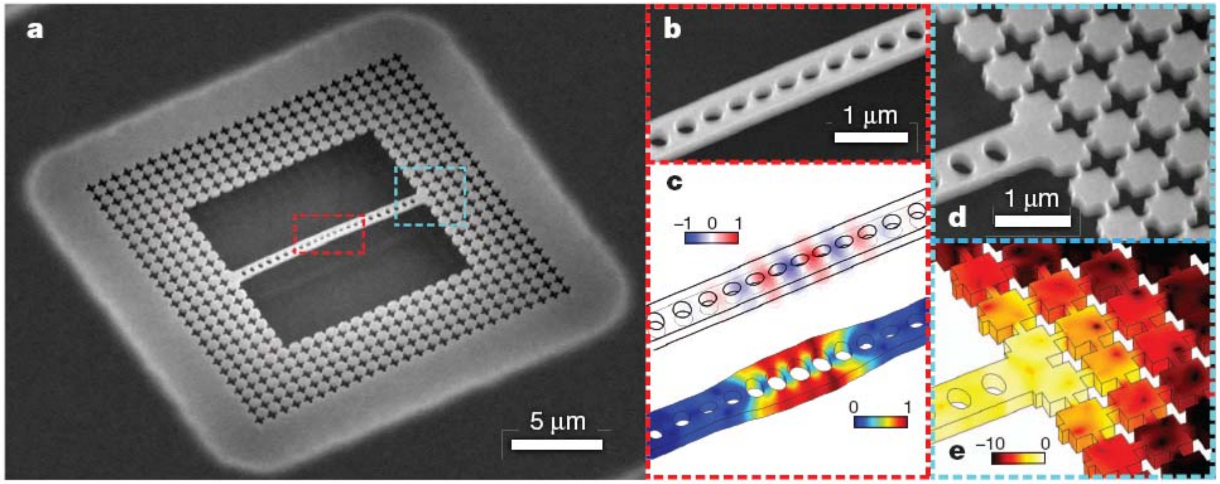
T=0.6K
(12GHz)

Verhagen, Deleglise, Weis, Schliesser and Kippenberg, Nature 482, 66 (2012)

EPFL

QUANTUM OPTOMECHANICS(2)

T~20K
(400GHz)



Chan et al., Nature 478, 90 (2011)

CALTECH, U. VIENNA

ORDERS OF MAGNITUDE

Group	ω_m	Q_m $=\omega_m/\gamma$	X_{ZPF}	ω_e	Q_e $=\omega_e/\kappa$	g_3	ω_m/κ
JILA/NIST K. Lehnert, R. Simmonds	10.56 MHz	3.3×10^5	4.1fm	7.54 GHz	4×10^4	0.5 kHz	50
EPFL T. Kippenberg	78 MHz	34	~1fm	385 THz	5×10^7	3.4 kHz	11
Caltech/Vienna O. Painter, M. Aspelmeyer	3.68 GHz	10^5	3fm	195 THz	5×10^5	500 kHz	7

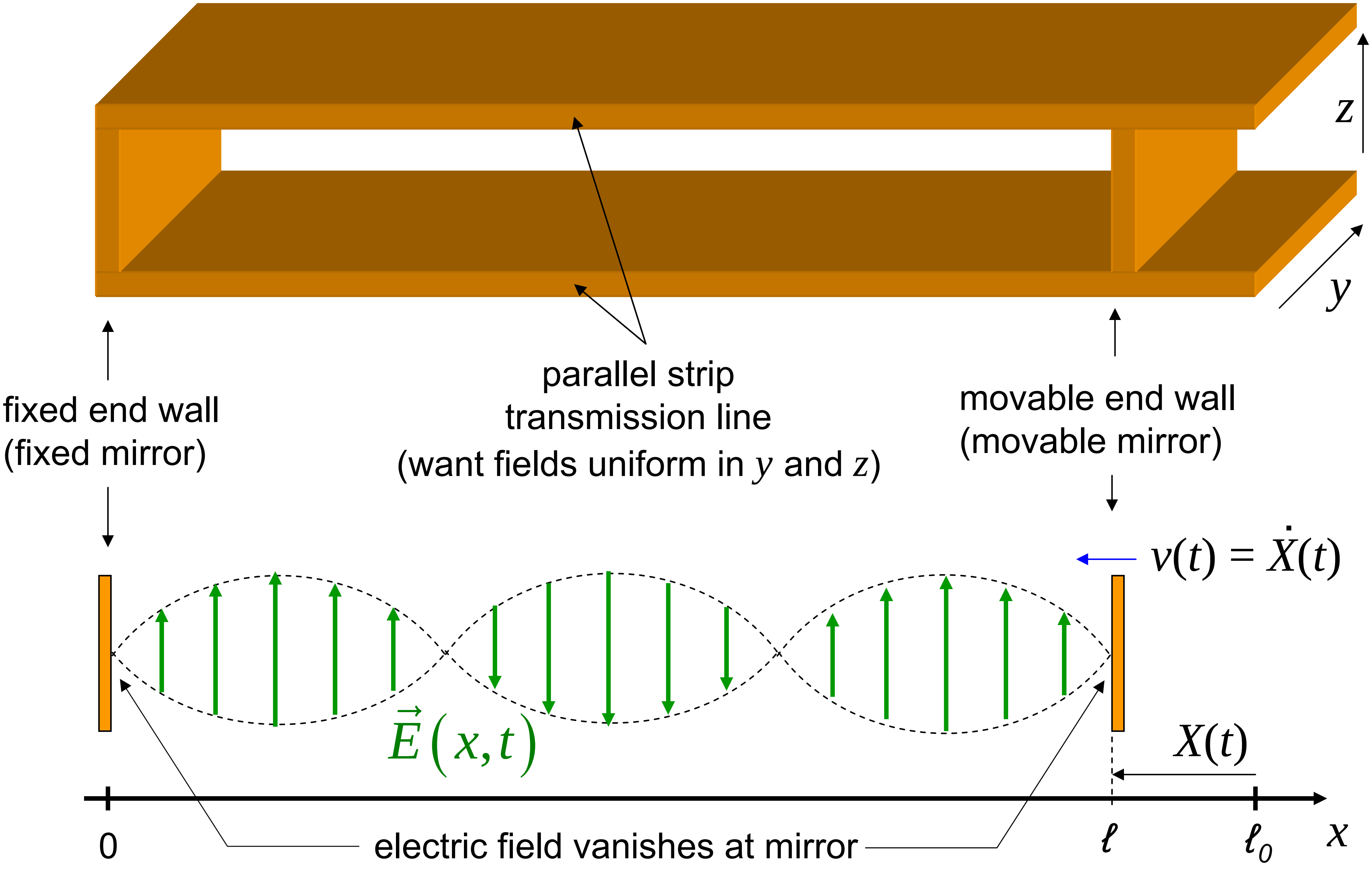
HOW IS THE HAMILTONIAN ABOVE, ALLOWING TO MEASURE fm DISPLACEMENTS, CONNECTED TO THE ONE RESPONSIBLE FOR THE DOPPLER MEASUREMENT OF VELOCITY AND THE ATTRACTION BETWEEN TWO PLATES SUBMITTED TO AN AC VOLTAGE?

WHAT APPROXIMATIONS ARE MADE IN THIS SIMPLE HAMILTONIAN?

THE CONNECTION BETWEEN RADIATION PRESSURE AND THE DOPPLER EFFECT CAN BE EXPLORED USING A SIMPLE MODEL, WHICH WILL ANSWER THESE QUESTIONS.

Reference: C. K. Law, Phys. Rev. **A 51,** 2537 (1995)

AN ELEMENTARY, DEFORMABLE 1D RESONATOR



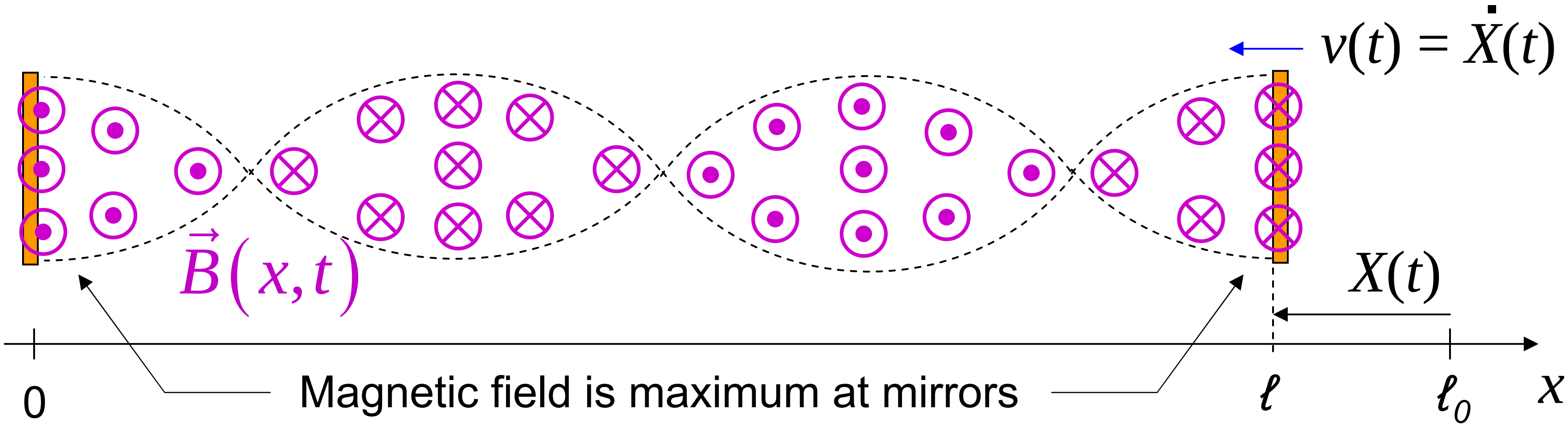
fixed end wall
(fixed mirror)

parallel strip
transmission line
(want fields uniform in y and z)

movable end wall
(movable mirror)

electric field vanishes at mirror

A SIMPLE, DEFORMABLE 1D RESONATOR (CNT'D)



Treat vector potential in radiation gauge (a.k.a. Coulomb or transverse gauge)

$$\vec{A}(x,t) = A(x,t)\,\hat{z}$$

The mirrors are perfectly conducting and inside the cavity, no sources are present. Thus:

$$A(\ell(t),t) = A(0,t) = 0 \qquad \vec{E}(x,t) = -\frac{\partial A(x,t)}{\partial t}\hat{z} \qquad \vec{B}(x,t) = -\frac{\partial A(x,t)}{\partial x}\hat{y}$$

The vector potential obeys:

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] A(x,t) = 0$$

speed of light

ENERGY INSIDE RESONATOR

Instantaneous energy density inside resonator:

$$\mathcal{E}(x,t) = \frac{\varepsilon_0}{2}\vec{E}(x,t)^2 + \frac{1}{2\mu_0}\vec{B}(x,t)^2$$

$$= \frac{\varepsilon_0}{2}\left[\frac{\partial}{\partial t}A(x,t)\right]^2 + \frac{1}{2\mu_0}\left[\frac{\partial}{\partial x}A(x,t)\right]^2$$

Instantaneous total energy inside resonator:

$$\mathcal{U} = \underset{\substack{\uparrow \\ \text{mirror area}}}{S}\int_0^{\ell}\mathcal{E}(x,t)\,dx = S\int_0^{\ell}\left\{\frac{\varepsilon_0}{2}\left[\frac{\partial}{\partial t}A(x,t)\right]^2 + \frac{1}{2\mu_0}\left[\frac{\partial}{\partial x}A(x,t)\right]^2\right\}dx$$

Crucial details are that ℓ is an implicit function of time through the dynamics of X

and that at all times $A(\ell(t),t) = A(0,t) = 0$

RADIATION PRESSURE

Pressure = derivative of energy inside an enclosure with respect to volume of enclosure

Thus, pressure on mirror is:

$$p = \frac{\partial \mathcal{U}}{\partial \mathcal{V}} = \left[\frac{\partial}{\partial \ell} \int_0^{\ell} \mathcal{E}(x,t)\, dx \right]$$

$$= \mathcal{E}(\ell, t)$$

$$= \frac{1}{2\mu_0} \left[\frac{\partial}{\partial x} A(x,t) \right]^2 \Bigg|_{\substack{x=\ell \\ A(\ell=0,t)=0}}$$

$$= \frac{1}{2\mu_0} \left[B(x,t) \right]^2 \Bigg|_{\substack{x=\ell \\ A(\ell=0,t)=0}}$$

Only space derivative contributes.
Time derivative of A always zero at mirror.

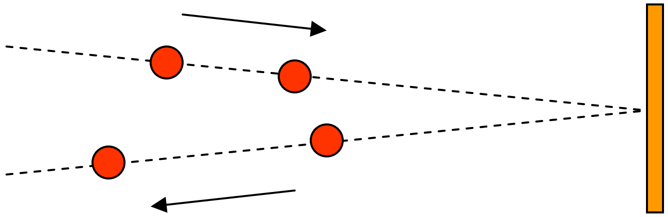
Pressure is given by square of magnetic field at mirror.

This remains correct even if ℓ varies with time.

Coupling between a macroscopic scatterer and light is fundamentally non-linear.

PHOTON APPROACH TO RADIATION PRESSURE

Photons each
have energy E
momentum E/c



Photons bounce on mirror

Momentum transfer to mirror: 2E/c per photon
(when perfect back-scattering)

Pressure: momentum transfer per unit area per unit time

Radiation pressure: 2 × incoming energy flux / c

$$p = 2\frac{\left|\vec{S}_{in}\right|}{c} = 2\frac{\left|\vec{E}_{in} \times \vec{B}_{in}\right|}{\mu_0 c}$$

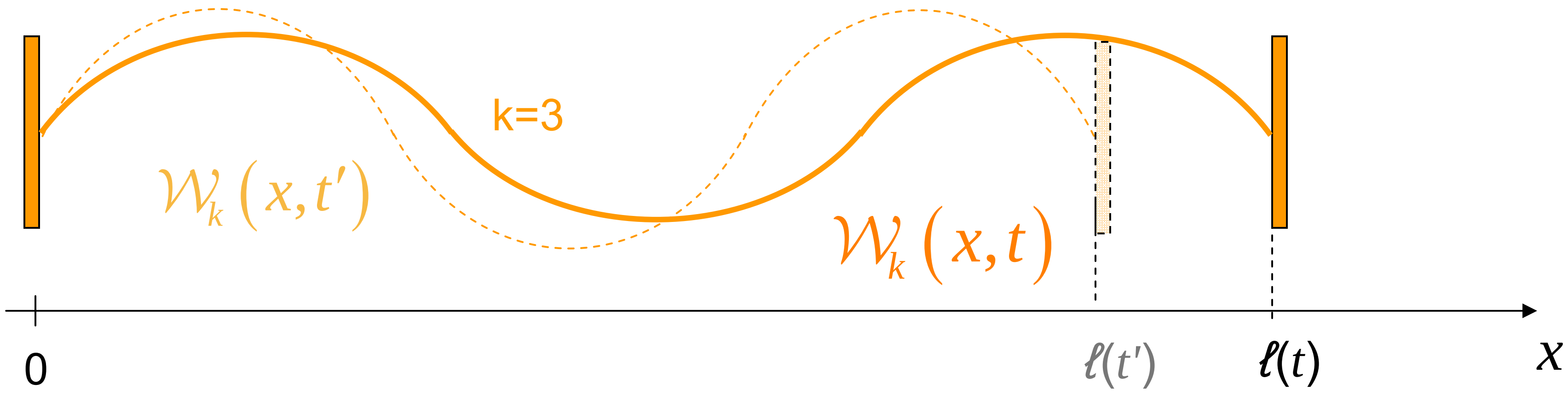
Poynting vector

TEM mode in cavity

$$= 2\frac{\left|\vec{B}_{in}\right|^2}{\mu_0} = \frac{\left|\vec{B}_{mirror}\right|^2}{2\mu_0}$$

Confirms wave approach!

NORMAL MODES OF A CAVITY WHOSE SHAPE DEPENDS ON TIME



$$\mathcal{W}_k(x,t)=\sqrt{\frac{2}{\ell(t)}}\sin\frac{\pi k x}{\ell(t)}$$

Both wavelength and amplitude are time dependent.
Symbol k : integer.

$$\boxed{\int_0^{\ell(t)}\mathcal{W}_k(x,t)\,\mathcal{W}_{k'}(x,t)=\delta_{k,k'}}$$

Orthonormality condition is independent of time.

(scheme is not relativistic, OK here)

Mode frequency:

$$\boxed{\omega_k\left[\ell(t)\right]=\frac{\pi c k}{\ell(t)}}$$

Also varies with time!

GENERALIZED NORMAL COORDINATES OF FIELD INSIDE CAVITY

Normal coordinates

$$q_k(t) = \int_0^{\ell(t)} A(x,t)\,\mathcal{W}_k(x,t)\,dx$$

Countable infinity of degrees of freedom.

In turn, the field can be expressed in terms of normal coordinates:

$$A(x,t) = \sum_{k=1}^{\infty} q_k(t)\,\mathcal{W}_k(x,t)$$

Completeness
of mode functions

Dimensional analysis:

$$[q] = [\underset{\substack{\uparrow \\ \text{flux}}}{\Phi}]\cdot[\ell]^{1/2}$$

CLASSICAL EQUATIONS OF MOTION FOR FIELD NORMAL COORDINATES AND MIRROR POSITION

Field equation: $c^2 \frac{\partial^2}{\partial x^2} A(x,t) = \frac{\partial^2}{\partial t^2} A(x,t)$

$$\omega_k^2 q_k = -\ddot{q}_k \overbrace{+ \frac{2\,\dot{\ell}}{\ell} \sum_j g_{kj} \dot{q}_j - \frac{\dot{\ell}^2 - \ddot{\ell}\ell}{\ell^2} \sum_j g_{kj} q_j + \frac{\dot{\ell}^2}{\ell^2} \sum_{j,l} g_{jk} g_{jl} q_l}$$

Newton's Law for mirror:

$$M\ \ddot{\ell} = \underset{\substack{\uparrow \\ \text{mech}^{\text{al}} \\ \text{force}}}{-\frac{\partial V(\ell)}{\partial \ell}} + \underbrace{\frac{1}{\mu_0 c^2 \ell} \sum_{k,j} (-1)^{k+j} \omega_k \omega_j q_k q_j}_{\text{radiation pressure force}}$$

coupled equations:

light presses on mirror, mirror pumps energy into or out of light.

where: $\omega_k(\ell) = \frac{\pi c k}{\ell}$ and: $g_{kj} = \begin{cases} (-1)^{k+j} \frac{2kj}{j^2 - k^2}, & k \neq j \\ 0, & k = j \end{cases}$

DOPPLER EFFECT

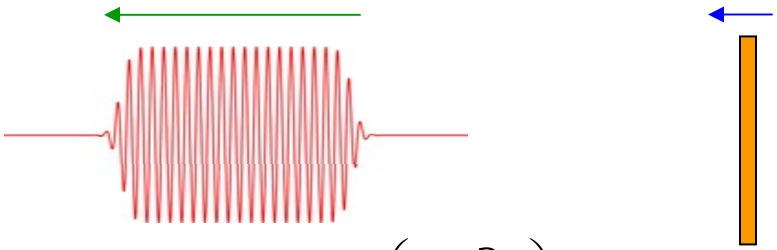
fixed mirror far away

$v = \text{constant}$

$v \ll c$

wave packet, velocity c
& frequency f

After reflection:

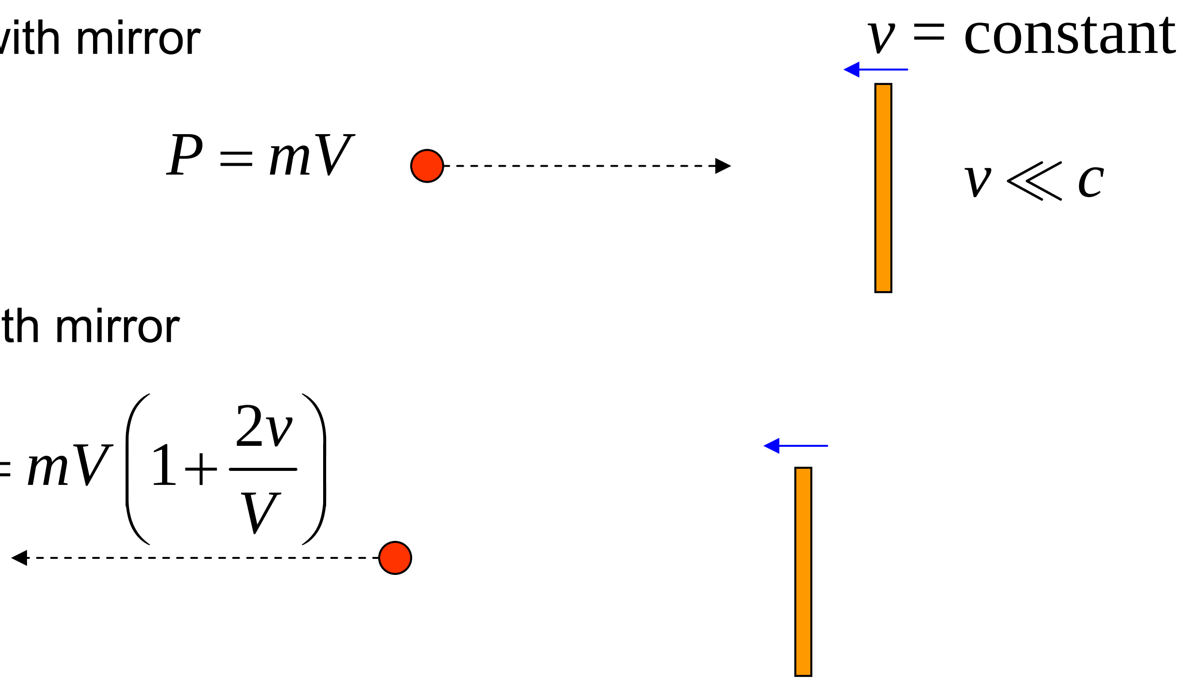


frequency is now $f\left(1+\frac{2v}{c}\right)$

energy in pulse has increased by same ratio

DOPPLER EFFECT FROM PHOTON POINT OF VIEW

Process is so far completely classical but can be interpreted in particle language:



before collision with mirror

$v = \text{constant}$

$$P = mV$$

$v \ll c$

after collision with mirror

$$P' = mV\left(1 + \frac{2v}{V}\right)$$

Particle bouncing elastically on a moving heavy wall sees its momentum P increased by $2mv$, where m is particle mass and v the wall velocity. Relative increase in P is thus $(1+ 2v/V)$, where V is the velocity of the particle. We can apply this idea to the photon:

$$P = \frac{\hbar\omega}{c} \rightarrow P' = \frac{\hbar\omega}{c}\left(1 + \frac{2v}{c}\right)$$

FULL HAMILTONIAN OF CAVITY WITH MOVING MIRROR IN THE NON-RELATIVISTIC APPROX$^{\text{TION}}$

From the classical equations, it is possible to construct a Lagrangian. One obtains the conjugate momenta and the quantum Hamiltonian by imposing standard commutation relation between coordinates and momenta.

$$\hat{H} = \frac{\left(\hat{p}+\hat{\Pi}\right)^2}{2M} + V\left(\hat{\ell}\right) + \hbar\sum_{k=1}^{\infty}\omega_k\left(\hat{\ell}\right)\hat{a}_k^\dagger\hat{a}_k - \frac{\pi\hbar c}{\hat{\ell}}$$

Casimir interaction (1-D)

where $$\hat{\Pi} = \frac{i\hbar}{2\hat{\ell}}\sum_{k,j}\left[g_{kj}\left(\frac{k}{j}\right)^{1/2}\left(\hat{a}_k^\dagger\hat{a}_j^\dagger - \hat{a}_k\hat{a}_j + \hat{a}_k^\dagger\hat{a}_j - \hat{a}_j^\dagger\hat{a}_k\right)\right]$$

and $$\left[\hat{a}_j\left(\hat{\ell}\right), \hat{a}_k^\dagger\left(\hat{\ell}\right)\right] = \delta_{jk}$$

Hamiltonian contains Doppler shift, radiation pressure and Casimir effect.
First term resembles usual gauge coupling of charged particle. But here, mirror is neutral.

Motion is non-relativistic and bounded by condition : $\ell > 0$.

TWO APPROXIMATIONS

If $\hat{X} = \hat{\ell} - \ell_0 \ll \ell_0$ Very well obeyed for nanoresonators in quantum regime!
(small displacement condition)

Then, after a gauge transformation: $\hat{H}_1 = \hat{T}^\dagger \hat{H} \hat{T}; \quad \hat{T} = \exp\left(i \hat{\Pi} \hat{X} \right)$

includes Casimir force

$$\hat{H}_1 \cong \frac{\hat{P}^2}{2M} + U\left(\hat{X}\right) + \hbar \sum_{k=1}^{\infty} \omega_{k0} \hat{a}_{k0}^\dagger \hat{a}_{k0} - \hat{X}\hat{F}_0$$

radiation pressure

$$\hat{F}_0 = \frac{\hbar}{2\ell_0} \sum_{k,j} (-1)^{k+j} \sqrt{\omega_{k0}\omega_{j0}} \left(\hat{a}_{k0}\hat{a}_{j0} + \hat{a}_{k0}^\dagger \hat{a}_{j0}^\dagger + \hat{a}_{k0}^\dagger \hat{a}_{j0} + \hat{a}_{j0}^\dagger \hat{a}_{k0} \right)$$

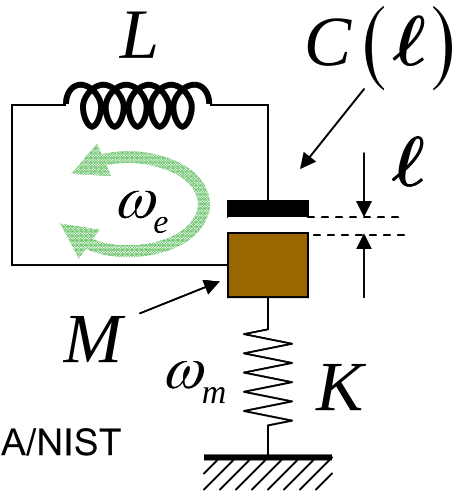
Harmonic restoring force for mirror: $U\left(\hat{X}\right) = \frac{K\hat{X}^2}{2}$

If $\omega_m = \sqrt{\frac{K}{M}} \ll \left| \omega_{k0} - \omega_{j0} \right|, \forall (k, j)$ Adiabatic condition

For a strictly 1D resonator, it corresponds to period of mirror motion much slower than roundtrip of photon.

we finally arrive at: $\hat{H}_1 \cong \frac{\hat{P}^2}{2M} + \frac{K\hat{X}^2}{2} + \hbar\omega_e \left(1 + \frac{\hat{X}}{\ell_0} \hat{a}^\dagger \hat{a} \right)$ QED!

ELECTROSTATIC PRESSURE



JILA/NIST

$$X = \ell - \ell_0 \ll \ell_0$$

$$C = \varepsilon \frac{S}{\ell} \cong \varepsilon \frac{S}{\ell_0}\left(1 - \frac{X}{\ell_0}\right)$$

$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C_0}\left(1 - \frac{\hat{X}}{\ell_0}\right) + \frac{\hat{P}^2}{2M} + \frac{K\hat{X}^2}{2}$$

$$\omega_e = \frac{1}{\sqrt{LC_0}}; \quad Z_e = \sqrt{\frac{L}{C_0}}$$

$$Q_{ZPF} = \sqrt{\frac{\hbar}{2Z_e}}$$

$$\hat{Q} = Q_{ZPF}\left(a + a^\dagger\right); \quad \hat{X} = X_{ZPF}\left(b + b^\dagger\right)$$

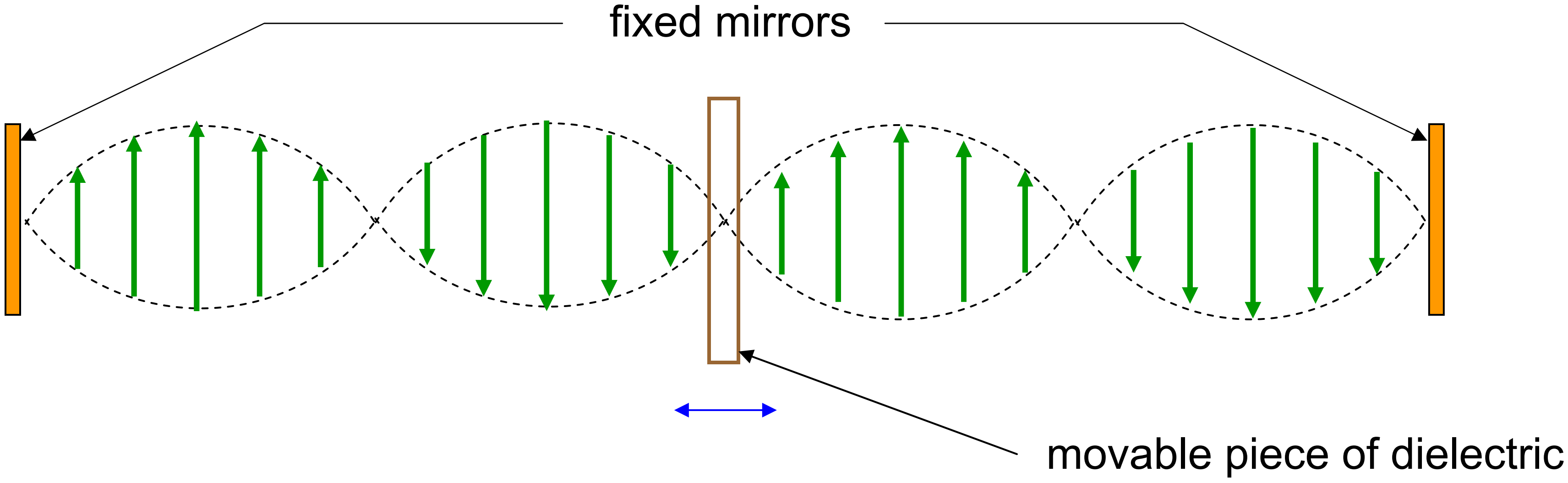
$$\frac{\hat{H}}{\hbar} = \frac{\omega_e}{4}\left[-\left(a - a^\dagger\right)^2 + \left(a + a^\dagger\right)^2\left(1 - \frac{X_{ZPF}}{\ell_0}\left(b + b^\dagger\right)\right)\right] + \omega_m b^\dagger b$$

Since $\omega_e \gg \omega_m$
can neglect terms
in $a^{\dagger 2}, a^2$

$$\frac{\hat{H}}{\hbar} \cong \omega_e a^\dagger a\left(1 - \frac{X_{ZPF}}{2\ell_0}\left(b + b^\dagger\right)\right) + \omega_m b^\dagger b$$

4-WAVE NON-LINEARITY

Sankey,Yang, Zwickl, Jayich, and Harris, Nature Physics **6**, 707 (2010)



Symmetry excludes from coupling hamiltonian terms odd in mechanical position

$$\frac{\hat{H}}{\hbar} = \omega_e \hat{a}^\dagger \hat{a} + \omega_m \hat{b}^\dagger \hat{b} + g_4 \left(\hat{b} + \hat{b}^\dagger\right)^2 \hat{a}^\dagger \hat{a}$$

Can be used to read phonon number directly.

$$g_4 = \left(X_{ZPF}\right)^2 \frac{\partial^2 \omega_e}{\partial \ell^2}$$

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END OF LECTURE