COLLÈGE DE FRANCE
1530

Chaire de Physique Mésoscopique
Michel Devoret
Année 2011, 10 mai - 21 juin

AMPLIFICATION ET RETROACTION QUANTIQUES

QUANTUM AMPLIFICATION AND FEEDBACK

Seconde Leçon / *Second Lecture*

Transparents des leçons disponibles à http://www.physinfo.fr/lectures.html

PROGRAM OF THIS YEAR'S LECTURES

Lecture I: Introduction to quantum-limited amplification and feedback

Lecture II: How do we model open, out-of-equilibrium, non-linear quantum systems?

Lecture III: Is it possible to optimize the parametric amplifier characteristics while maintaining its noise at the quantum limit?

Lecture IV: What are the minimal requirements for an active circuit to be fully directional and noiseless?

Lecture V: Can continuous quantum measurements be viewed as a form of Brownian motion?

Lecture VI: How can we maintain a dynamic quantum state alive?

Please note that there will be no lecture on May 24

CALENDAR OF SEMINARS

May 10: Fabien Portier, SPEC-CEA Saclay
The Bright Side of Coulomb Blockade

May 17, 2011: Jan van Ruitenbeek (Leiden University, The Netherlands)
Quantum Transport in Single-molecule Systems

May 31, 2011: Irfan Siddiqi (UC Berkeley, USA)
Quantum Jumps of a Superconducting Artificial Atom

June 7, 2011: David DiVicenzo (IQI Aachen, Germany)
Quantum Error Correction and the Future of Solid State Qubits

June 14, 2011: Andrew Cleland (UC Santa Barbara, USA)
Images of Quantum Light

June 21, 2011: Benjamin Huard (LPA - ENS Paris)
Building a Quantum Limited Amplifier from Josephson Junctions and Resonators

June 21, 2011 (3pm): Andrew Cleland (UC Santa Barbara, USA)
How to Be in Two Places at the Same Time ?

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LECTURE II : MODELLING OPEN, OUT-OF-EQUILIBRIUM, NON-LINEAR QUANTUM CIRCUITS

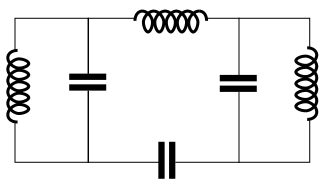
OUTLINE

1. Modes of isolated, linear quantum circuits, non-linear processes
2. Open, out-of-equilibrium, linear systems: input-output theory
3. Characterizing non-linear elements, participation ratio

11-II-4

CLOSED, NON-DISSIPATIVE LINEAR CIRCUITS

Example:

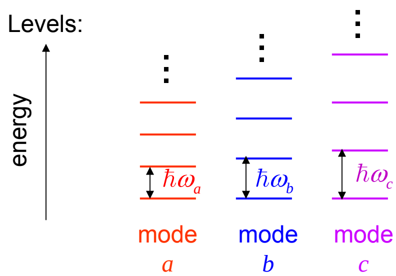


- just inductances and capacitances
- no sources and no resistances

→ undamped normal modes

\# of modes = # independent pairs of capacitances and inductances

Levels:



Hamiltonian:

$$H = \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar\omega_c c^\dagger c$$
$$= \sum_{m=a,b,c} \hbar\omega_m a_m^\dagger a_m$$

m : normal mode index

a_m : normal mode amplitude

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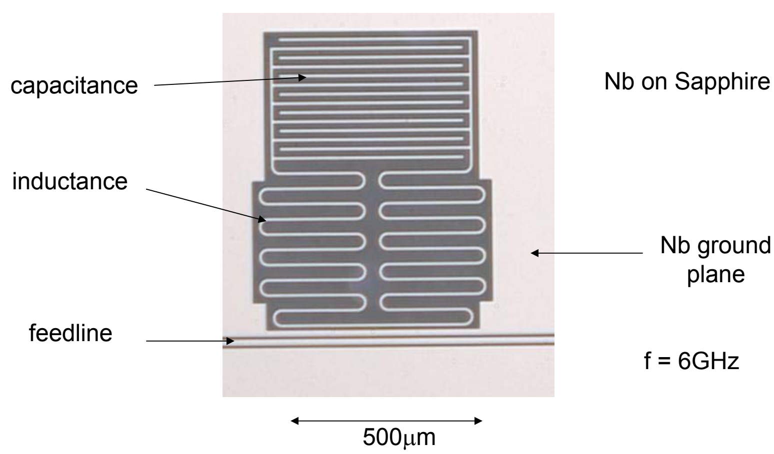
LUMPED ELEMENTS MICROWAVE CIRCUITS



Nb on Sapphire

f = 6GHz

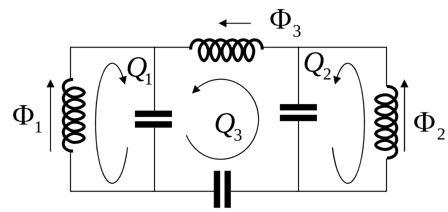
11-II-6

HAMILTONIAN FROM CHARGES AND FLUXES

Use loop variables:

$$\Phi_i^{loop} = \int_{-\infty}^{t} V_i^{ind}\, dt_1$$

$$Q_i^{loop} = \int_{-\infty}^{t} I_i^{cap}\, dt_1$$



$$\left[\Phi_i^{loop}, Q_i^{loop}\right] = i\hbar$$

Hamiltonian:

$$H = \frac{\Phi_1^2}{2L_1} + \frac{\Phi_2^2}{2L_2} + \frac{\Phi_3^2}{2L_3} + \frac{\left(Q_1 + Q_3\right)^2}{2C_1} + \frac{\left(Q_2 - Q_3\right)^2}{2C_2} + \frac{Q_3^2}{2C_3}$$

Inverse-Capacitance matrix:

$$\mathcal{C}_{ij}^{-1} = \frac{\partial^2 H}{\partial Q_i \partial Q_j} \qquad \dot{\Phi}_i = V_i = \sum_j \mathcal{C}_{ij}^{-1} Q_j$$

Inverse-Inductance matrix:

$$\mathcal{L}_{ij}^{-1} = \frac{\partial^2 H}{\partial \Phi_i \partial \Phi_j} \qquad \dot{Q}_i = I_i = -\sum_j \mathcal{L}_{ij}^{-1} \Phi_j$$

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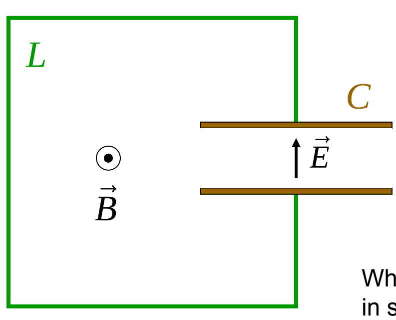
COMMUTATION RELATION OF FLUXES AND CHARGES



E.M. Lagrangian:

$$\iiint_{vol} \frac{dv}{2}\left(\varepsilon_0 E^2 - \mu_0^{-1} B^2\right)$$

$$= \iiint_{vol} \frac{dv}{2}\left(\varepsilon_0 \left(\frac{\partial}{\partial t}\vec{A}\right)^2 - \mu_0^{-1}\left(\vec{\nabla} \times \vec{A}\right)^2\right)$$

vector potential
= field amplitude

When magnetic and electric fields do not coexist in space (lumped elements), hamiltonian of corresponding mode is given by:

$$H = \frac{Q_{cap}^2}{2C} + \frac{\Phi_{ind}^2}{2L}$$

11-II-8

CURRENTS AND VOLTAGES OF NORMAL MODES

Equation of motions in matrix form:

$$V = \dot{\Phi} = \mathcal{C}^{-1}Q$$

$$I = \dot{Q} = -\mathcal{L}^{-1}\Phi$$

positive symmetric

Matrix of eigenfrequencies

$$\Omega = \left(\mathcal{L}^{-1/2}\mathcal{C}^{-1}\mathcal{L}^{-1/2}\right)^{1/2}$$

$$\Omega = O\begin{pmatrix} \omega_a & 0 & 0 \\ 0 & \omega_b & 0 \\ 0 & 0 & \omega_c \end{pmatrix}O^{-1}$$

orthogonal

Impedance matrix

$$Z = \mathcal{L}^{+1/4}\mathcal{C}^{-1/2}\mathcal{L}^{+1/4}$$

$$Z = O^{-1}\begin{pmatrix} Z_a & 0 & 0 \\ 0 & Z_b & 0 \\ 0 & 0 & Z_c \end{pmatrix}O$$

$$\Phi = O\sqrt{\frac{\hbar Z}{2}}\left(a + a^{\dagger}\right)$$

$$Q = O\sqrt{\frac{\hbar}{2Z}}\left(\frac{a - a^{\dagger}}{i}\right)$$

column vectors

1 photon
@ f=10GHz
Z =100Ω

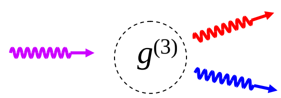
⇩

δV ~ 1μV
δI ~ 10nA

11-II-9

AMPLIFICATION EXPLOITS COMBINATION OF ANHARMONICITY, DRIVE AND DAMPING

$$H = \hbar\omega_a a^{\dagger}a + \hbar\omega_b b^{\dagger}b + \hbar\omega_c c^{\dagger}c + \left[\hbar g^{(3)} a^{\dagger}b^{\dagger}c + \text{h.c.}\right]$$

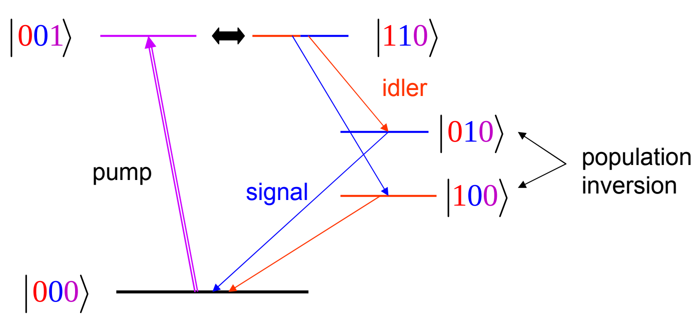


"3-wave mixing"

$$g^{(3)} \ll \omega_a, \omega_b, \omega_c$$

resonance condition:

$$\omega_c = \omega_a + \omega_b$$



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UNDERSTANDING OF AMPLICATION INVOLVES JOINT TREATMENT OF:

1. STEADY-STATE OUT-OF-EQUILIBRIUM

2. NON-LINEAR PROCESSES

11-II-11

OUTLINE

1. Modes of isolated, linear quantum circuits, non-linear processes
2. Open, out-of-equilibrium, linear systems: input-output theory
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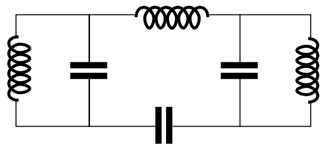
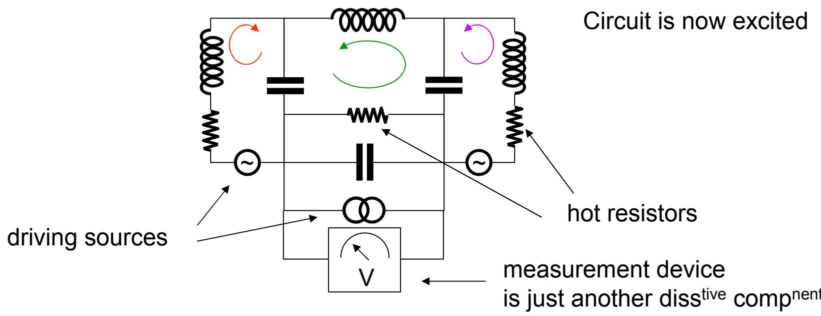
11-II-4

OPEN, DRIVEN, DISSIPATIVE LINEAR CIRCUITS

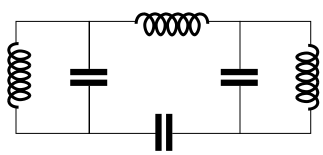


Couple circuit to out-of-equilibrium environment

Circuit is now excited

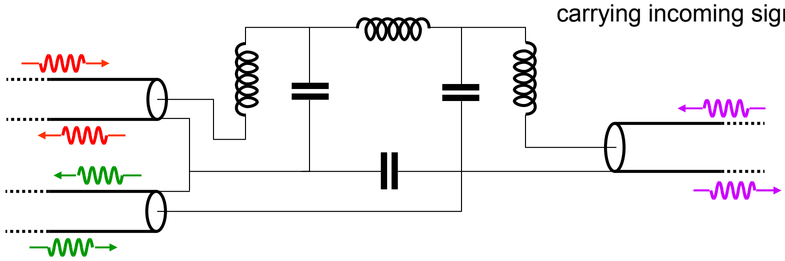


OPEN, DISSIPATIVE LINEAR CIRCUITS



Couple circuit to out-of-equilibrium environment

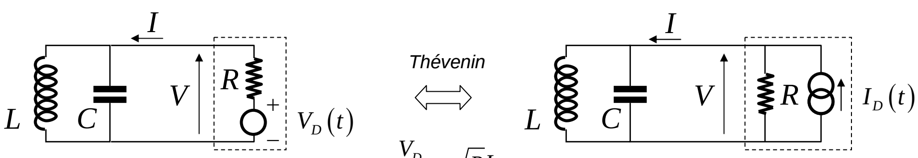
Environment is modeled as semi-infinite transmission lines carrying incoming signals



System loses energy by radiating outgoing signals into transmission lines, reaching steady state equilibrium.

CORRESPONDANCE BETWEEN DRIVE AND INPUT FIELDS, DISSIPATION AND LINE IMPEDANCE

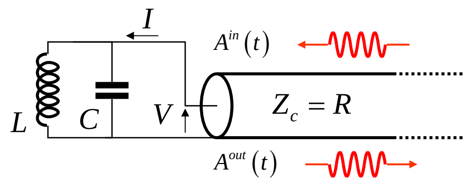
Dissipative drive being equivalent to semi-infinite transmission line with incoming signals, system dynamics is described by relation between input and output fields



$V = V_D - RI$ $\qquad \frac{V_D}{\sqrt{R}} = \sqrt{R} I_D$ $\qquad I = I_D - V/R$

$$A^{in}(t) = \frac{V}{2\sqrt{R}} + \frac{\sqrt{R}I}{2} = \frac{V_D}{2\sqrt{R}} = \frac{\sqrt{R}I_D}{2}$$

$$A^{out}(t) = \frac{V}{2\sqrt{R}} - \frac{\sqrt{R}I}{2} = \frac{V}{\sqrt{R}} - A^{in}(t) = A^{in}(t) - \sqrt{R}I$$



Information carried by outgoing signal corresponds to that acquired by a volt- or an ammeter

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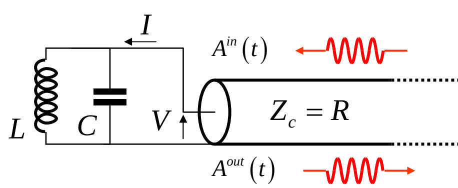
PRINCIPLE OF INPUT-OUTPUT CALCULATIONS



Hamiltonian of circuit → functional relation between I and V: $f\{I(t), V(t)\} = 0$

Here, for the LC circuit, we obtain: $\frac{d}{dt} I(t) = \left(C \frac{d^2}{dt^2} + \frac{1}{L} \right) V(t)$

In this relation, we now operate the substitution:

$$V = \sqrt{R}\left[A^{in} + A^{out}\right]$$
$$I = \frac{1}{\sqrt{R}}\left[A^{in} - A^{out}\right]$$

Expressing the outgoing field in terms of the incoming fields, we obtain the input-output relations.

For our example: $C\ddot{A}^{out} + \frac{1}{R}\dot{A}^{out} + \frac{1}{L}A^{out} = -C\ddot{A}^{in} + \frac{1}{R}\dot{A}^{in} - \frac{1}{L}A^{in}$

11-II-15

FOURIER TRANSFORMS OF FIELDS

Starting from time-domain expression $C\ddot{A}^{out} + \frac{1}{R}\dot{A}^{out} + \frac{1}{L}A^{out} = -C\ddot{A}^{in} + \frac{1}{R}\dot{A}^{in} - \frac{1}{L}A^{in}$

and introducing Fourier transforms $A^{in/out}[\omega] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\omega t} A^{in/out}(t)\,dt$

We obtain: $A^{iout}[\omega] = \left(\frac{-y(\omega)+1}{y(\omega)+1}\right) A^{in}[\omega]$ $\qquad y(\omega) = R\left(iC\omega + \frac{1}{iL\omega}\right) = i\frac{\omega^2-\omega_0^2}{\gamma\,\omega}$

unity modulus

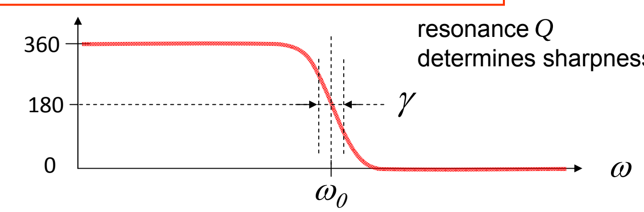
reduced admittance
(purely imaginary)

resonance
frequency

damping
rate

When resonance quality factor $Q = \frac{\omega_0}{\gamma} \gg 1$ RWA $y(\omega) = i\frac{\omega-\omega_0}{\gamma/2}$

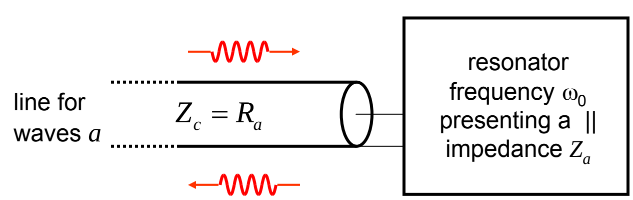
phase shift (deg)



resonance Q
determines sharpness

11-II-15

PHYSICAL MEANING OF DAMPING RATE IN INPUT-OUTPUT PICTURE



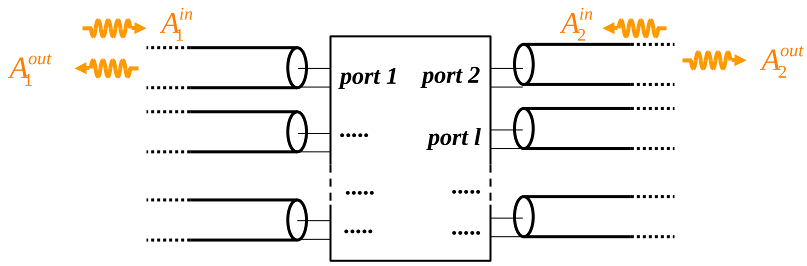
Propagating photons @ frequency ω_0 enter the resonator and reside a time γ^{-1} as standing waves before being re-radiated back.

$$\gamma = \frac{R_a}{Z_a}\omega_0 = R_a Y_a \omega_0$$

environment resonator

11-II-16

SCATTERING MATRIX FOR A LINEAR SYSTEM



$$\begin{bmatrix} A_1^{out}[\omega] \\ A_2^{out}[\omega] \\ \cdots \\ A_l^{out}[\omega] \\ \cdots \end{bmatrix} = \begin{bmatrix} s_{11}[\omega] & s_{12}[\omega] & \cdots & s_{1l}[\omega] & \cdots \\ s_{21}[\omega] & s_{22}[\omega] & \cdots & s_{2l}[\omega] & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{l1}[\omega] & s_{l2}[\omega] & \cdots & s_{ll}[\omega] & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} A_1^{in}[\omega] \\ A_2^{in}[\omega] \\ \cdots \\ A_l^{in}[\omega] \\ \cdots \end{bmatrix}$$

all compo-nents are at same frequency!

will change for amplifier

OUTGOING AMPLITUDES

"S" PARAMETERS, SCATTERING MATRIX

INCOMING AMPLITUDES

11-II-17

WAVE AMPLITUDE vs PHOTON OPERATORS

operator of wave amplitude in Fourier domain

propagation direction

angular frequency (positive or negative)

$$\boxed{A^{in,out}[\omega] = \sqrt{\frac{\hbar|\omega|}{2}}\, a^{in,out}[\omega]}$$

field ladder operator

$a[-|\omega|] = a[|\omega|]^{\dagger}$

$[A[\omega]]$ = [power $^{1/2}$ × time] = [action]$^{1/2}$

$[a[\omega]]$ = [time]$^{1/2}$

Ladder operators have commutation relations:

$$\left[a^{in,out}[\omega_1], a^{in,out}[\omega_2]\right] = \mathrm{sgn}(\omega_1 - \omega_2)\,\delta(\omega_1 + \omega_2)$$

Scattering always preserves commutation relations

11-II-18

NUMBER OF PHOTONS IN INPUT SIGNAL

Wave amplitude spectral density:

$$\langle A[\omega_1]A[\omega_2]\rangle = S_{AA}[\omega_1]\,\delta(\omega_1+\omega_2)$$

In T equilibrium: $S_{A^{in}A^{in}}[\omega] = \frac{\hbar\omega}{4}\left[\coth\left(\frac{\hbar\omega}{2k_BT}\right)+1\right]$

$\left[S_{AA}[\omega]\right]$ = [energy]

$$S_{A^{in}A^{in}}[\omega] \xrightarrow[T\to\infty]{} \frac{k_BT}{2}$$

Photon amplitude spectral density:

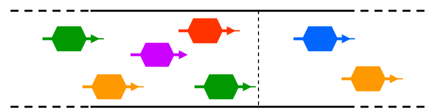
$$\langle a[\omega_1]a[\omega_2]\rangle = S_{aa}[\omega_1]\,\delta(\omega_1+\omega_2)$$

In thermal equilibrium:

$$\begin{cases} N^T_{a^{in}}(|\omega|) = \frac{1}{2}\coth\left(\frac{\hbar|\omega|}{2k_BT}\right) \\ N^T_{a^{in}}(|\omega|) \xrightarrow[T\to\infty]{} \frac{k_BT}{\hbar|\omega|} \\ N^T_{a^{in}}(|\omega|) \xrightarrow[T\to 0]{} \frac{1}{2} \end{cases}$$

Number of photons per unit time per unit bandwidth crossing a section of line:

$$N_a(|\omega|) = S_{aa}[+|\omega|] + S_{aa}[-|\omega|]$$



N.B.: Link between "engineer" and "physicist" spectral densities

standard frequency in Hz

$$\mathcal{S}_{XX}[\nu] = S_{XX}[\omega = 2\pi\nu] + S_{XX}[\omega = -2\pi\nu]$$

SCATTERING MATRIX FOR PASSIVE LINEAR CIRCUIT

When circuit contains only linear capacitances and inductances:

$$\mathrm{S}[\omega] = \left(\mathrm{Z}_c^{1/2}\mathrm{Y}[\omega]\mathrm{Z}_c^{1/2}+1\right)^{-1}\left(-\mathrm{Z}_c^{1/2}\mathrm{Y}[\omega]\mathrm{Z}_c^{1/2}+1\right)$$

Admittance matrix of circuit
Gives current in port k as function of voltages in port l. Can be computed directly from hamiltonian.

Diagonal matrix of line impedances

This expression is a generalization of the formula for reflection on a load:

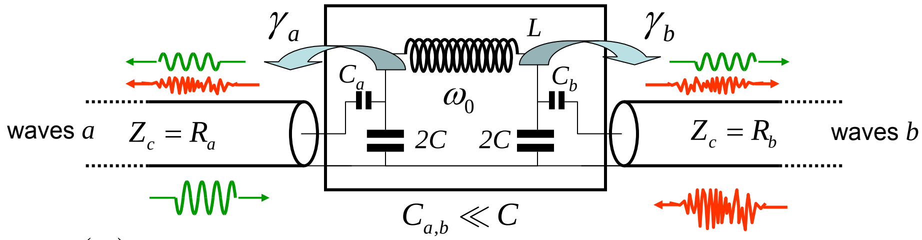
$$r = \frac{Z_L - Z_c}{Z_L + Z_c}$$

Y matrix is i times a positive hermitian matrix $\Rightarrow$ unitarity of S matrix $\boxed{\mathrm{S}^\dagger\mathrm{S}=1}$ $\left\{\begin{array}{l}\text{information conservation}\\ \text{energy conservation}\end{array}\right.$

Another property, fully general:

causality $\Rightarrow$ POLES OF S MATRIX IN LOWER-HALF PLANE

EXAMPLE: INPUT-OUTPUT TREATMENT OF DISPERSIVE REFLECTION ATTENUATOR



$s_{aa}(\omega)$ — attenuation of reflected signal

$$S = \left[\begin{array}{c|c} \dfrac{\Gamma_a - \Gamma_b - i\Delta\omega}{\Gamma_a + \Gamma_b + i\Delta\omega} & -\dfrac{2\Gamma_a^{1/2}\Gamma_b^{1/2}}{\Gamma_a + \Gamma_b + i\Delta\omega} \\ \hline -\dfrac{2\Gamma_a^{1/2}\Gamma_b^{1/2}}{\Gamma_a + \Gamma_b + i\Delta\omega} & \dfrac{\Gamma_b - \Gamma_a - i\Delta\omega}{\Gamma_a + \Gamma_b + i\Delta\omega} \end{array}\right]$$

$$\Gamma_{a,b} = \frac{\gamma_{a,b}}{2} \cong \frac{R_{a,b} C_{a,b}^2}{4LC^2}$$

$$\Delta\omega = \omega - \omega_0$$

Interference: when $\Delta\omega = 0$ & $\gamma_a = \gamma_b$, reflection vanishes!

Added noise by attenuator in reflection of a :
(referred to input)

$$\mathcal{A} = N_b \frac{|s_{ab}|^2}{|s_{aa}|^2} = N_b \left(|s_{aa}|^{-2} - 1 \right)$$

11-II-21

OUTLINE

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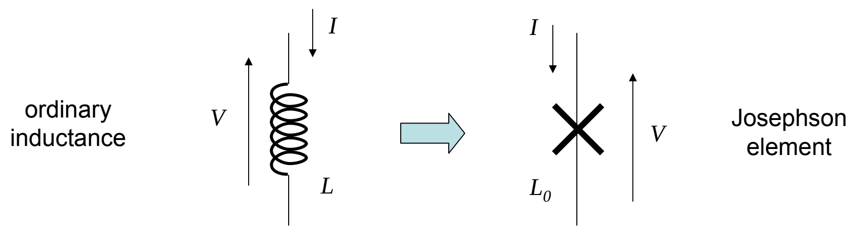
11-II-4

JE = NON-DISSIPATIVE, NON-LINEAR INDUCTANCE



$$V(t) = L\frac{dI(t)}{dt} \qquad V(t) = \phi_0 \frac{d}{dt}\sin^{-1}\frac{I(t)}{I_0} \qquad \phi_0 = \frac{\hbar}{2e}$$

$$= \frac{\phi_0}{I_0}\frac{1}{\sqrt{1-\frac{I(t)^2}{I_0^2}}}\frac{dI(t)}{dt}$$

current-dependence of inductance →

Energy stored in Josephson element: $E(t) = \int_0^{I(t)} V(t)\,dI = E_J\left(1-\cos\left[\sin^{-1}\frac{I(t)}{I_0}\right]\right)$

$$E_J = \phi_0 I_0$$

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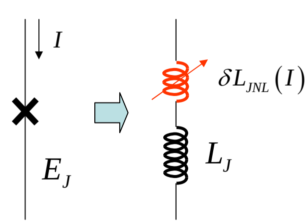
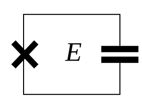
CHARACTERIZING NON-LINEARITY



$$L_J = \frac{\hbar^2}{(2e)^2 E_J}$$

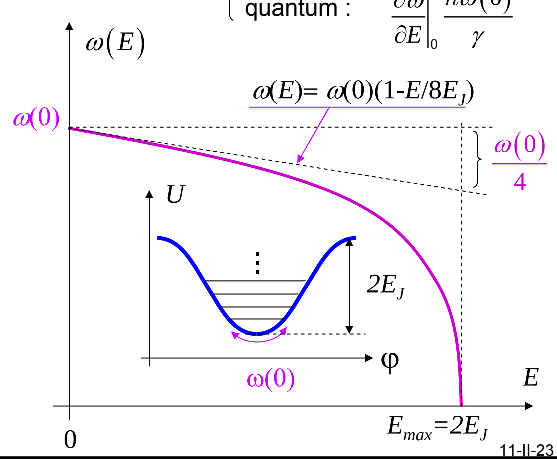
$$\delta L_{JNL} = L_J\left[\left(1-\frac{I^2}{I_0^2}\right)^{-1/2} - 1\right]$$

$$= \frac{1}{2}L_J\frac{I^2}{I_0^2} + O\left(\frac{I^4}{I_0^4}\right)$$



$$\omega(E) = \sqrt{\frac{1}{L(E)C}}$$

Anharmonicity
- classical : $\left.\frac{\partial\omega}{\partial E}\right|_0 \frac{E_{max}}{\omega(0)} = \frac{1}{4}$
- quantum : $\left.\frac{\partial\omega}{\partial E}\right|_0 \frac{\hbar\omega(0)}{\gamma}$



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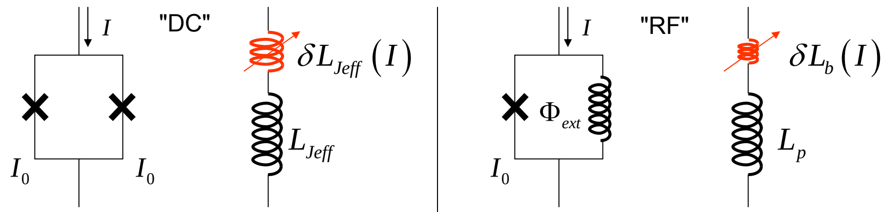
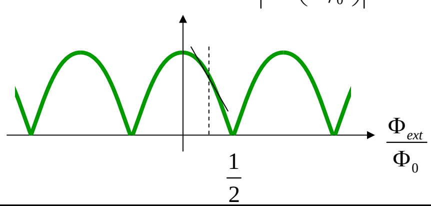
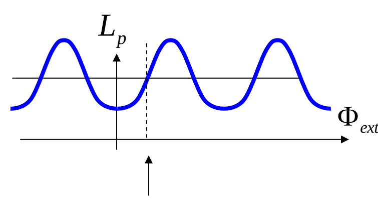
SQUID's: MODULATION OF NON-LINEARITY



$$L(I) = \frac{\phi_0}{I_{Jeff}}\left[1 + \frac{1}{2}\frac{I^2}{I_{Jeff}^2} + O\left(\frac{I^4}{I_0^4}\right)\right]$$

$$I_{Jeff} = 2I_0\left|\cos\left(\frac{\Phi_{ext}}{2\phi_0}\right)\right|$$





non-linearity maximum here

see Benjamin Huard's seminar

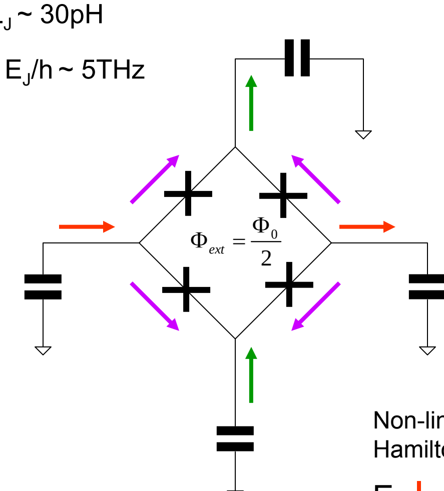
11-II-24

EVALUATION OF $g^{(3)}$

I_0 ~ 10µA ⇨ L_J ~ 30pH

⇨ E_J/h ~ 5THz

f ~ 5GHz, Z ~ 1Ω



1 photon ~ 100nA
~ $I_0/100$

Non-linear term in
Hamiltonian is of order:

$$\frac{E_J\, I_X I_Y I_Z}{(I_0)^3} \Rightarrow g^{(3)} \sim \text{MHz}$$

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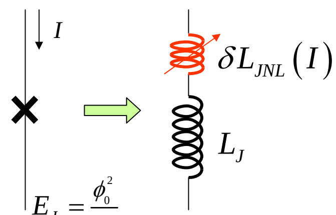
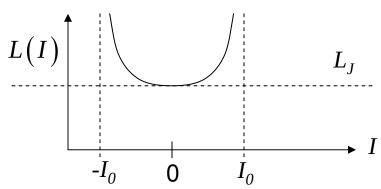
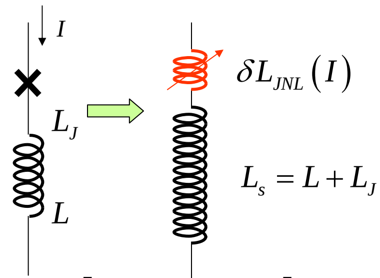
DILUTION OF NON-LINEARITY



$$E_J = \frac{\phi_0^2}{L_J}$$

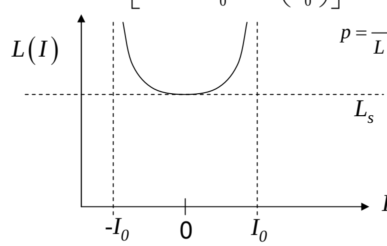
$$L(I) = L_J\left[1 + \frac{1}{2}\frac{I^2}{I_0^2} + O\left(\frac{I^4}{I_0^4}\right)\right]$$





$$L(I) = L_s\left[1 + \frac{1}{2}p\frac{I^2}{I_0^2} + O'\left(\frac{I^4}{I_0^4}\right)\right]$$

$$p = \frac{L_J}{L + L_J}$$



decrease in participation ratio

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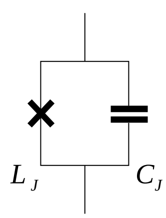
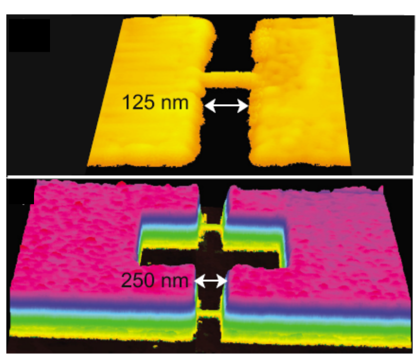
BANDWIDTH OF NON-LINEARITY



$$\omega_p = \sqrt{\frac{1}{L_J C_J}}$$

Plasma frequency is ultimate bandwidth limitation (20-30GHz)

Nanobridges have a higher plasma frequency than tunnel junctions and are promising non-linear elements for superconducting amplifiers.



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END OF LECTURE