COLLÈGE DE FRANCE 1530

Chaire de Physique Mésoscopique
Michel Devoret
Année 2008, 13 mai - 24 juin

CIRCUITS ET SIGNAUX QUANTIQUES

QUANTUM SIGNALS AND CIRCUITS

Cinquième Leçon / *Fifth Lecture*

08-V-1

VISIT THE WEBSITE OF THE CHAIR OF MESOSCOPIC PHYSICS

http://www.college-de-france.fr

and follow links to:

http://www.physinfo.fr/lectures.html

PDF FILES OF ALL LECTURES WILL BE POSTED ON THIS WEBSITE

Questions, comments and corrections are welcome!

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CALENDAR OF SEMINARS

May 13: Denis Vion, (Quantronics group, SPEC-CEA Saclay)
Continuous dispersive quantum measurement of an electrical circuit

May 20: Bertrand Reulet (LPS Orsay)
Current fluctuations : beyond noise

June 3: Gilles Montambaux (LPS Orsay)
Quantum interferences in disordered systems

June 10: Patrice Roche (SPEC-CEA Saclay)
Determination of the coherence length in the Integer Quantum Hall regime

June 17: Olivier Buisson, (CRTBT-Grenoble)
A quantum circuit with several energy levels

June 24: Jérôme Lesueur (ESPCI)
High Tc Josephson Nanojunctions: Physics and Applications

08-V-3

PROGRAM OF THIS YEAR'S LECTURES

Lecture I: Introduction and overview

Lecture II: Modes of a circuit and propagation of signals

Lecture III: The "atoms" of signal

Lecture IV: Quantum fluctuations in transmission lines

Lecture V: Introduction to non-linear active circuits

Lecture VI: Amplifying quantum signals with dispersive circuits

08-V-4

LECTURE V : INTRODUCTION TO NON-LINEAR ACTIVE CIRCUITS

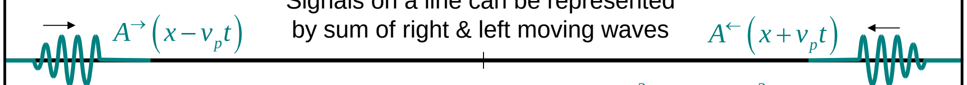
<u>OUTLINE</u>

1. Where we stand so far, purpose of this lecture
2. Fluctuations of damped harmonic oscillator
3. Hamiltonian approach of Caldeira and Leggett
4. Properties of scattering matrix
5. Non-linear active circuits

08-V-5

1ST QUANTIZA^TION^ OF SIGNALS (CLASS^ICAL^ WAVES)

Signals on a line can be represented by sum of right & left moving waves



Energy flux @ $x = 0$: $\mathcal{P}(t) = \left|A^{\rightarrow}(t)\right|^2 - \left|A^{\leftarrow}(t)\right|^2$

Decomposition of signal into modes:

$$A_{mp}^{\rightleftarrows} = \int_{-\infty}^{+\infty} dt\, \psi_{mp}^{*}(t) A^{\rightleftarrows}(t)$$

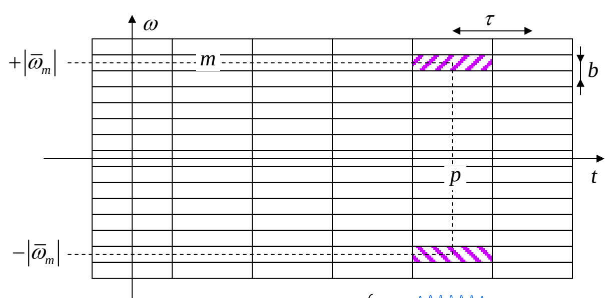
$$A_{-mp}^{\rightleftarrows} = \left(A_{mp}^{\rightleftarrows}\right)^{*}$$

$$\left[A_{mp}^{\rightleftarrows}\right] = [\text{energy}]^{1/2}$$

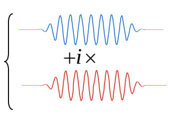
Each mode is a "flying oscillator":

$$\left\{A_{m_1 p_1}^{\rightleftarrows}, A_{m_2 p_2}^{\rightleftarrows}\right\}_{\text{P.B.}} = -\frac{i}{2}\bar{\omega}_{m_1} \delta_{m_1+m_2} \delta_{p_1-p_2}$$

INFORMATION OF OSCILLATORS IS CONSERVED



$\psi_{mp}(t)$



"cos"

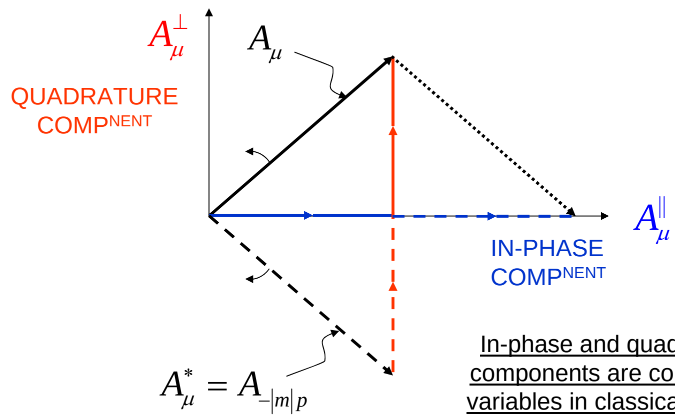
"sin"

$$\int_{-\infty}^{+\infty} dt\, \psi_{m_1 p_1}^{*}(t) \psi_{m_2 p_2}(t) = \delta_{m_1 m_2} \delta_{p_1 p_2}$$

08-V-6c

1ST QUANTIZATION OF SIGNALS: FRESNEL REPRESENTATION OF MODE

mode index $\mu = \{(|m|, p), \rightarrow \text{ or } \leftarrow\}$ refers to a pair of tiles



In-phase and quadrature components are conjugate variables in classical sense

08-V-7

2ND QUANTIZATION OF SIGNALS (QUANTUM WAVES)

classical mode amplitude $A_{mp}^{\rightleftarrows} \rightarrow \hat{A}_{mp}^{\rightleftarrows}$ quantum operator

Poisson bracket $\left\{A_{m_1 p_1}^{\rightleftarrows}, A_{m_2 p_2}^{\rightleftarrows}\right\}_{P.B.} \rightarrow \left[\hat{A}_{m_1 p_1}^{\rightleftarrows}, \hat{A}_{m_2 p_2}^{\rightleftarrows}\right] = \frac{\hbar \varpi_{m_1}}{2} \delta_{m_1+m_2} \delta_{p_1-p_2}$ commutator

physical observables

continuous field variables

continuous field ladder operators: $\hat{a}[\omega = \omega_m] = \lim_{b \to 0} \frac{1}{\sqrt{b \frac{\hbar |\omega|_m}{2}}} \hat{A}_{mp}$

(N.B.: arrows have been dropped)

commutator of a's: $\left[\hat{a}[\omega_1], \hat{a}[\omega_2]\right] = \mathrm{sg}(\omega_1 - \omega_2)\, \delta[\omega_1 + \omega_2]$

average value of anticommutator in thermal state: $\left\langle \left\{\hat{a}[\omega_1], \hat{a}[\omega_2]\right\} \right\rangle_T = \coth \frac{\hbar |\omega_1|}{2 k_B T} \delta[\omega_1 + \omega_2]$

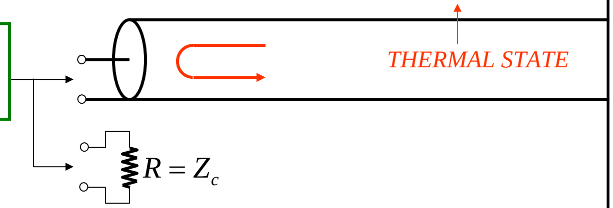
08-V-8a

QUANTUM FLUCTUATION-DISSIPATION THEOREM

$$\left\langle \hat{V}^{\rightleftarrows}[\omega_1]\hat{V}^{\rightleftarrows}[\omega_2]\right\rangle = S_{VV}^{\rightleftarrows}[\omega_1]\,\delta(\omega_1+\omega_2)$$

$$S_{VV}^{\rightleftarrows}[\omega] = \frac{Z_c}{4}\hbar\omega\left[\coth\left(\frac{\hbar\omega}{2k_BT}\right)+1\right]$$

$$S_{VV}[\omega] = R\hbar\omega\left[\coth\left(\frac{\hbar\omega}{2k_BT}\right)+1\right]$$



$$S_{VV}[\omega] = 4S_{VV}^{\rightleftarrows}[\omega]$$

QUANTUM FLUCTUATION-DISSIPATION THEOREM

$$\left\langle \hat{V}^{\rightleftarrows}[\omega_1]\hat{V}^{\rightleftarrows}[\omega_2]\right\rangle = S_{VV}^{\rightleftarrows}[\omega_1]\,\delta(\omega_1+\omega_2)$$

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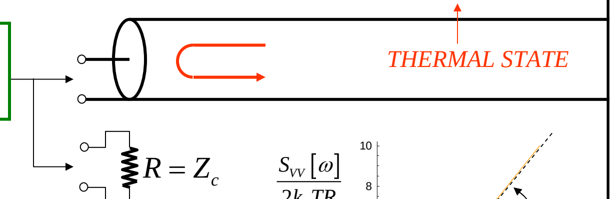
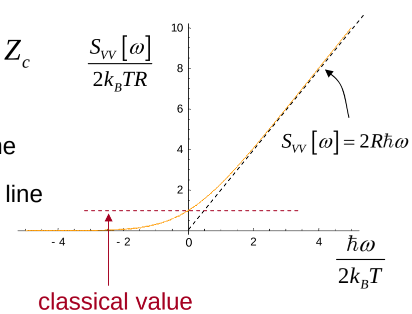


$$S_{VV}[\omega] = 4S_{VV}^{\rightleftarrows}[\omega]$$

QUANTUM ASYMMETRY OF FLUCTUATIONS

ω > 0 : emission into the line

ω < 0 : absorption from the line



QUANTUM FLUCTUATION-DISSIPATION THEOREM

$$\left\langle \hat{V}^{\rightleftarrows}[\omega_1]\hat{V}^{\rightleftarrows}[\omega_2]\right\rangle = S_{VV}^{\rightleftarrows}[\omega_1]\,\delta(\omega_1+\omega_2)$$

$$S_{VV}^{\rightleftarrows}[\omega] = \frac{Z_c}{4}\hbar\omega\left[\coth\left(\frac{\hbar\omega}{2k_BT}\right)+1\right]$$

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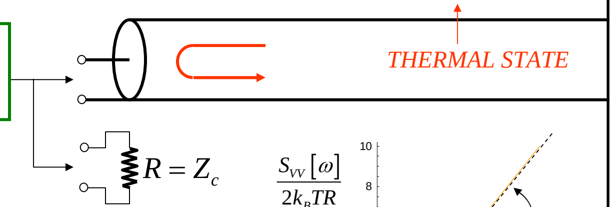
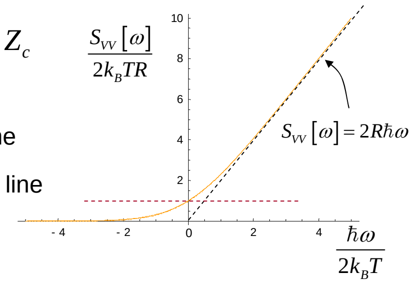


$$S_{VV}[\omega] = 4S_{VV}^{\rightleftarrows}[\omega]$$



QUANTUM ASYMMETRY OF FLUCTUATIONS

$\omega > 0$: emission into the line

$\omega < 0$: absorption from the line

Recover results of Johnson-Nyquist noise:

Voltage fluctuations in classical regime:

$$\mathcal{S}_{VV}[\nu] = 4k_BTR$$

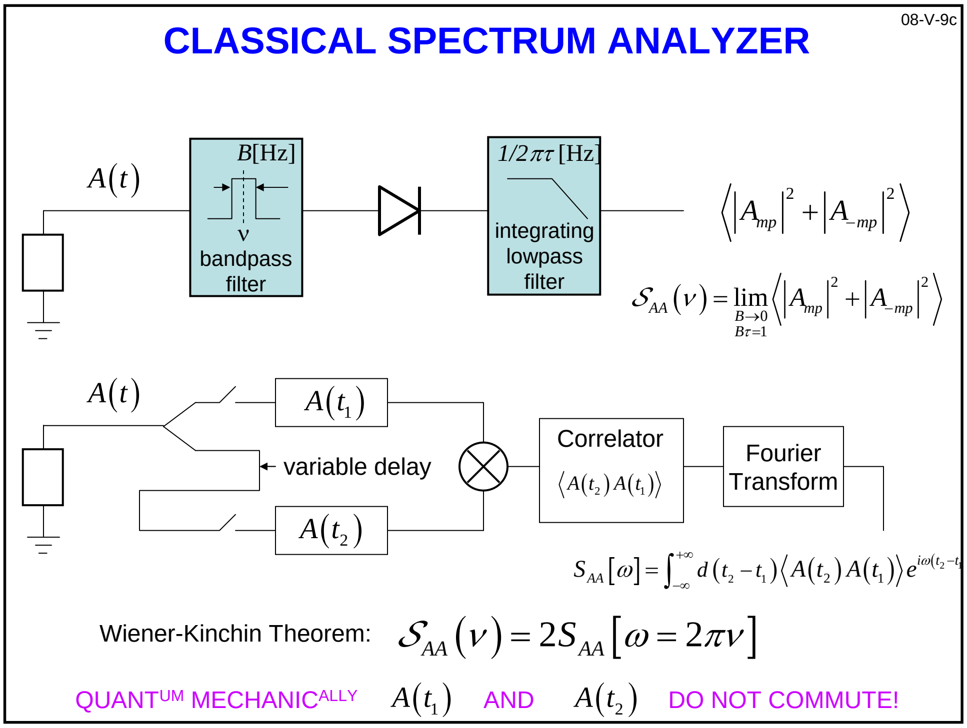
Line power flow in classical regime:

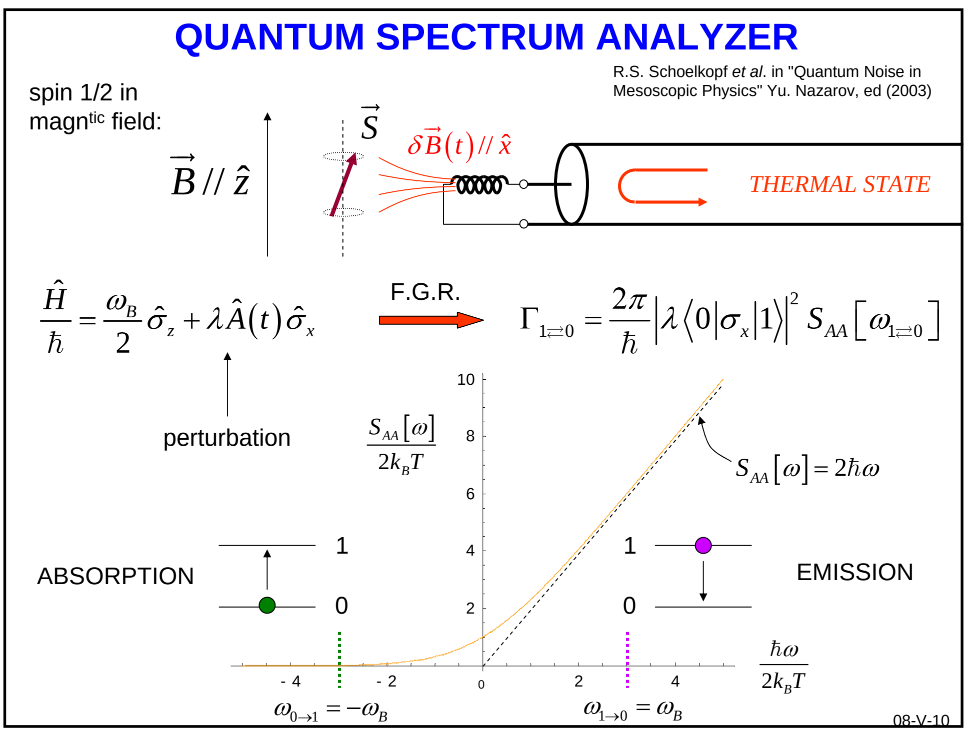
$$P = k_BTB$$ (k_BT per mode)

08-V-8c

CAN WE BUILD A
QUANTUM SPECTRUM ANALYZER?

08-V-8d


08-V-9c
CLASSICAL SPECTRUM ANALYZER
$A(t)$
B[Hz]
ν
bandpass filter
$1/2\pi\tau$ [Hz]
integrating lowpass filter
$\left\langle \left|A_{mp}\right|^2 + \left|A_{-mp}\right|^2 \right\rangle$
$\mathcal{S}_{AA}(\nu) = \lim_{\substack{B\to 0 \\ B\tau=1}} \left\langle \left|A_{mp}\right|^2 + \left|A_{-mp}\right|^2 \right\rangle$
$A(t)$
$A(t_1)$
variable delay
$A(t_2)$
Correlator
$\langle A(t_2) A(t_1) \rangle$
Fourier Transform
$S_{AA}[\omega] = \int_{-\infty}^{+\infty} d(t_2 - t_1) \langle A(t_2) A(t_1) \rangle e^{i\omega(t_2-t_1)}$
Wiener-Kinchin Theorem: $\mathcal{S}_{AA}(\nu) = 2S_{AA}[\omega = 2\pi\nu]$
QUANTUM MECHANICALLY $A(t_1)$ AND $A(t_2)$ DO NOT COMMUTE!


QUANTUM SPECTRUM ANALYZER
R.S. Schoelkopf et al. in "Quantum Noise in Mesoscopic Physics" Yu. Nazarov, ed (2003)
spin 1/2 in magntic field:
$\vec{B} // \hat{z}$
$\vec{S}$
$\delta\vec{B}(t) // \hat{x}$
THERMAL STATE
$\frac{\hat{H}}{\hbar} = \frac{\omega_B}{2}\hat{\sigma}_z + \lambda\hat{A}(t)\hat{\sigma}_x$
perturbation
F.G.R.
$\Gamma_{1\rightleftarrows 0} = \frac{2\pi}{\hbar}\left|\lambda\langle 0|\sigma_x|1\rangle\right|^2 S_{AA}[\omega_{1\rightleftarrows 0}]$
$\frac{S_{AA}[\omega]}{2k_BT}$
10
8
6
4
2
$S_{AA}[\omega] = 2\hbar\omega$
ABSORPTION
EMISSION
1
0
1
0
- 4
- 2
0
2
4
$\frac{\hbar\omega}{2k_BT}$
$\omega_{0\to 1} = -\omega_B$
$\omega_{1\to 0} = \omega_B$
08-V-10

PURPOSE OF THIS LECTURE

WHAT IS THE EFFECT OF THE QUANTUM PART OF FLUCTUATIONS?

DIFFERENT CLASSES OF RESPONSE OF CIRCUITS:

- DISPERSIVE vs DISSIPATIVE
- PASSIVE vs ACTIVE
- LINEAR vs NON-LINEAR

08-V-11

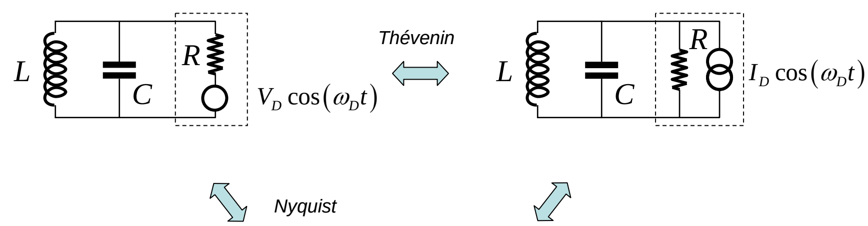
OUTLINE

1. Where we stand so far, purpose of this lecture
2. Fluctuations of damped harmonic oscillator
3. Hamiltonian approach of Caldeira and Leggett
4. Properties of scattering matrix
5. Non-linear active circuits

08-V-5b

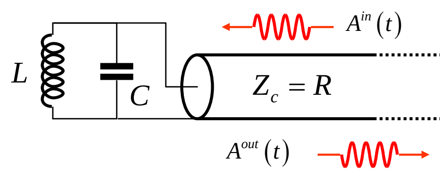
SCATTERING APPROACH TO DRIVEN DISSIPATIVE CIRCUITS

MAIN IDEA: RESISTANCE IS EQUIVALENT TO SEMI-INFINITE TRANSMISSION LINE



$A^{in}(t): A^{\leftarrow}(t)$

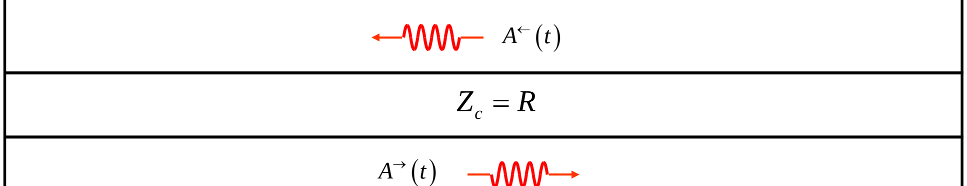
$A^{out}(t): A^{\rightarrow}(t)$



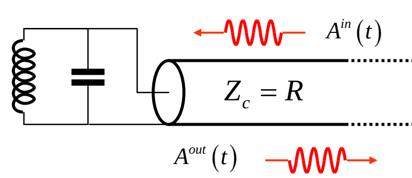
DRIVE CAN BE TREATED AS INCOMING SIGNAL

08-V-12a

A KEY IDEA



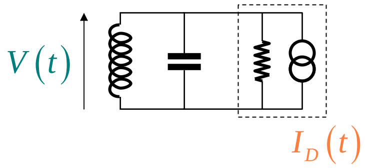
UNLIKE $A^{\leftarrow}(t)$ AND $A^{\rightarrow}(t)$, $A^{in}(t)$ AND $A^{out}(t)$ ARE NOT INDEPENDENT



TERMINATING CIRCUIT IMPOSES A STRICT RELATIONSHIP BETWEEN INCOMING AND OUTGOING WAVES

08-V-13

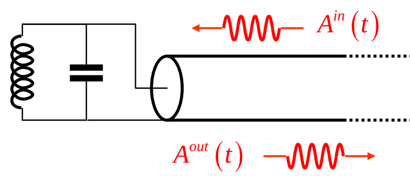
SAME CIRCUIT, DIFFERENT POINTS OF VIEW



DAMPED, DRIVEN STANDING OSCILLATOR

$I_D(t) \rightarrow V(t)$

IMPEDANCE



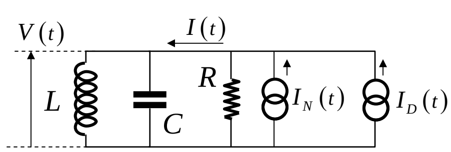
PURELY DISPERSIVE SCATTERER OF WAVES

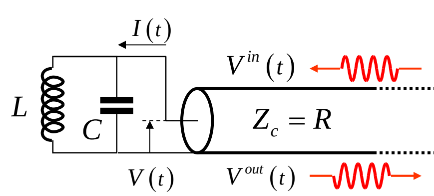
$A^{in}(t) \rightarrow A^{out}(t)$

SCATTERING MATRIX

08-V-14

RESISTANCE-TRANS$^{\text{MISSION}}$ LINE EQUIVALENCE





$$\begin{cases} V(t) = \dot{\Phi}(t) = R\left(I_D(t) + I_N(t) - I(t)\right) \\ C\ddot{\Phi} + \dfrac{\dot{\Phi}}{R} + \dfrac{\Phi}{L} = I_D(t) + I_N(t) \end{cases}$$

$$\ddot{\Phi} + \frac{\dot{\Phi}}{RC} + \frac{\Phi}{LC} = \frac{I_D(t) + I_N(t)}{C}$$

$$\boxed{V(t) + Z_c I(t) = 2V^{in}(t)}$$

$$\begin{cases} V(t) = V^{in}(t) + V^{out}(t) \\ V^{in/out}(t) = Z_c^{1/2}\, A^{in/out}(t) \\ I(t) = I^{in}(t) - I^{out}(t) \\ I^{in/out}(t) = Z_c^{-1/2}\, A^{in/out}(t) \end{cases}$$

$$\boxed{\ddot{\Phi} + 2\Gamma\dot{\Phi} + \omega_0^2 \Phi = 4\Gamma V^{in}(t)}$$

$$\boxed{V^{out}(t) = \dot{\Phi} - V^{in}(t)}$$

$\omega_0 = \dfrac{1}{\sqrt{LC}}$ resonance frequency

$\Gamma = \dfrac{1}{2RC}$ amplitude damping rate (HWHM)

$\dfrac{\omega_0}{2\Gamma} = Q \gg 1$ quality factor

08-V-15b

INPUT-OUTPUT RELATION IN OPERATOR FORM

$$\ddot{\Phi} + 2\Gamma\dot{\Phi} + \omega_0^2\Phi = 4\Gamma V^{in}(t)$$

$$V^{out}(t) = \dot{\Phi} - V^{in}(t)$$

$$\frac{d^2}{dt^2}\hat{\Phi} + 2\Gamma\frac{d}{dt}\hat{\Phi} + \omega_0^2\hat{\Phi} = 4\Gamma\hat{V}^{in}(t)$$

$$\hat{V}^{out}(t) = \frac{d}{dt}\hat{\Phi} - \hat{V}^{in}(t)$$

Fourier domain:

$$\left(\omega_0^2 - \omega^2 + 2i\omega\Gamma\right)\hat{\Phi}[\omega] = 4\Gamma\hat{V}^{in}[\omega]$$

$$\hat{V}^{out}[\omega] = i\omega\hat{\Phi}[\omega] - \hat{V}^{in}[\omega]$$

$$\hat{V}^{out}[\omega] = -\frac{\omega_0^2 - \omega^2 - 2i\omega\Gamma}{\omega_0^2 - \omega^2 + 2i\omega\Gamma}\hat{V}^{in}[\omega]$$

UNIT MODULUS COMPLEX NUMBER

08-V-16a

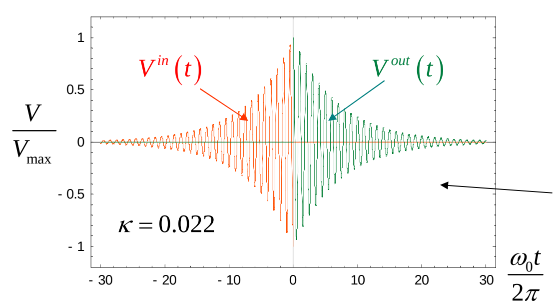
ROTATING WAVE APPROXIMATION

Since $\frac{\Gamma}{\omega_0} = \kappa = \frac{1}{2Q} \ll 1$ we can consider 1 pole at a time

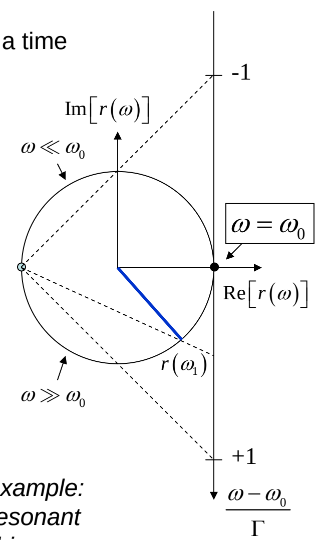
$$r(\omega) = -\frac{\omega_0^2 - \omega^2 - 2i\omega\Gamma}{\omega_0^2 - \omega^2 + 2i\omega\Gamma} \simeq \frac{1 - i\frac{\omega - \omega_0}{\Gamma}}{1 + i\frac{\omega - \omega_0}{\Gamma}}$$

$\omega > 0;\ r(-|\omega|) = r(|\omega|)^*$

$$V[\omega] = (1 + r[\omega])V^{in}[\omega] \simeq \frac{2}{1 + i\frac{\omega - sgn(\omega)\omega_0}{\Gamma}}V^{in}[\omega]$$



example: resonant drive



08-V-17c

FLUCTUATIONS OF THE RESISTIVELY DAMPED LC RESONATOR

$$\left\langle \hat{\Phi}[-\omega]\hat{\Phi}[\omega] \right\rangle = \frac{\left|1+r\left(|\omega|\right)\right|^2}{\omega^2} S_{VV}^{in}[\omega]$$

$$= Z_c \frac{\left|1+r\left(|\omega|\right)\right|^2}{\omega^2} \left(\frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2k_B T} \right)$$

$$\left\langle \hat{\Phi}^2 \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \left\langle \hat{\Phi}[-\omega]\hat{\Phi}[\omega] \right\rangle$$ converges

$$\left\langle \hat{Q}^2 \right\rangle = \frac{1}{2\pi Z_0^2} \int_{-\infty}^{+\infty} d\omega\, \omega^2 \left\langle \hat{\Phi}[-\omega]\hat{\Phi}[\omega] \right\rangle$$ **diverges !**

$$Z_0 = \sqrt{\frac{L}{C}}$$

08-V-18a

FLUX FLUCTUATIONS OF THE DAMPED LC RESONATOR

INTEGRAL CAN BE PERFORMED ANALYTICALLY AND YIELD MEANINGFUL RESULTS

H. Grabert, U. Weiss and P. Talkner, Z. Phys. B 55 (1984) 87

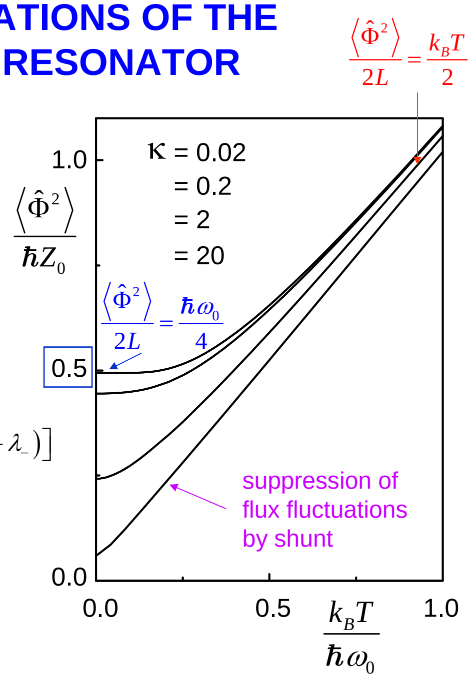
dimensionless fluctuations

dimensionless temperature

$$\frac{\left\langle \hat{\Phi}^2 \right\rangle}{\hbar Z_0} = \frac{k_B T}{\hbar\omega_0} + \frac{1}{2\pi\sqrt{\kappa^2-1}} \left[\Psi\left(1+\lambda_+\right) - \Psi\left(1+\lambda_-\right) \right]$$

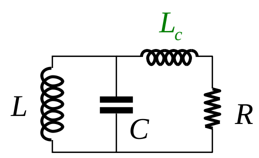
$\Psi(x)$ is polygamma function

$$\lambda_\pm = \frac{\kappa \pm \sqrt{\kappa^2-1}}{2\pi k_B T / \left(\hbar\omega_0\right)}$$



08-V-19a

CHARGE FLUCTUATIONS OF THE DAMPED LC RESONATOR WITH INDUCTIVE CUTOFF

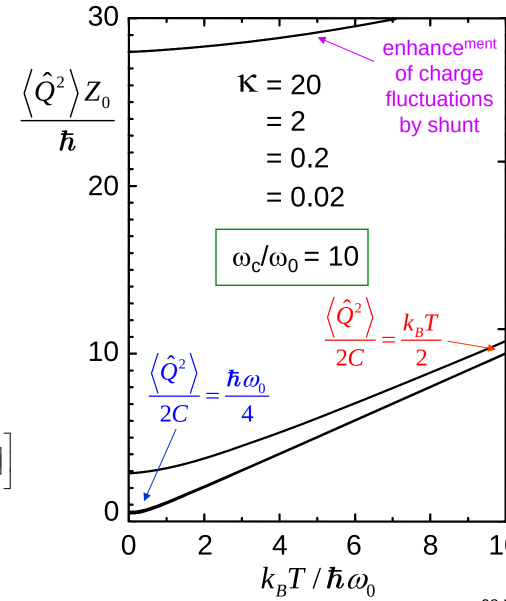


$$\omega_c = \frac{R}{L_c}$$

$$\left\langle \hat{Q}^2 \right\rangle = \frac{1}{Z_0^2}\left\langle \hat{\Phi}^2 \right\rangle + \Delta_c$$

$$\Delta_c = \frac{\hbar\kappa}{\pi Z_0}\left[2\Psi\left(1+\lambda_c\right) - \frac{1}{\sqrt{\kappa^2-1}}\left[\lambda_+\Psi\left(1+\lambda_+\right) - \lambda_-\Psi\left(1+\lambda_-\right)\right]\right]$$

$$\lambda_c = \frac{\hbar\omega_0}{2\pi k_B T}\left(\frac{\omega}{\omega_c} - 2\kappa\right)$$



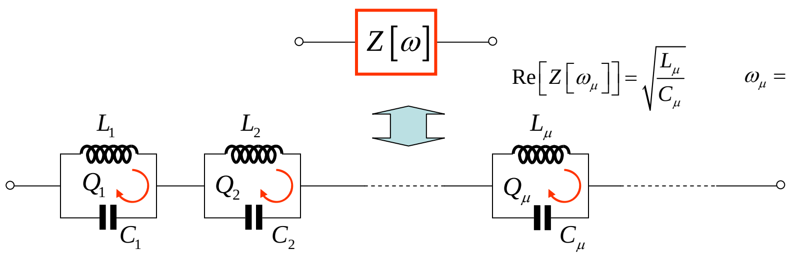
08-V-20a

OUTLINE

1. Where we stand so far, purpose of this lecture
2. Fluctuations of damped harmonic oscillator
3. Hamiltonian approach of Caldeira and Leggett
4. Properties of multiple ports scattering matrix
5. Non-linear active circuits

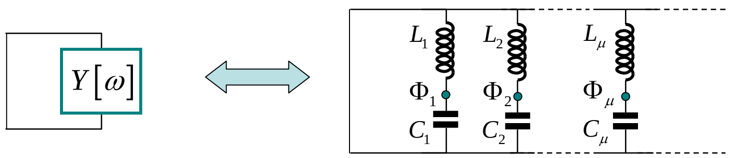
08-V-5c

ANOTHER POINT OF VIEW: HAMILTONIAN APPROACH OF CALDEIRA AND LEGGETT



$$\mathrm{Re}\left[Z\left[\omega_\mu\right]\right] = \sqrt{\frac{L_\mu}{C_\mu}} \qquad \omega_\mu = \sqrt{\frac{1}{L_\mu C_\mu}}$$

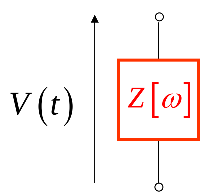
standing oscillator modes; nb. of oscillators $N \to \infty$



$$\mathrm{Re}\left[Y\left[\omega_\mu\right]\right] = \sqrt{\frac{C_\mu}{L_\mu}} \qquad \omega_\mu = \sqrt{\frac{1}{L_\mu C_\mu}}$$

08-V-21a

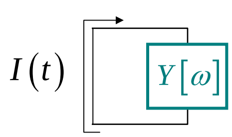
GENERALIZATION OF QUANTUM FLUCTUATION-DISSIPATION THEOREM TO ARBITRARY IMPEDANCE OR ADMITTANCE



$$S_{VV}[\omega] = \hbar\omega\left[\coth\left(\frac{\hbar\omega}{2k_B T}\right)+1\right]\mathrm{Re}\left(Z[\omega]\right)$$

QUANTUM BLACK-BODY BOSE FACTOR

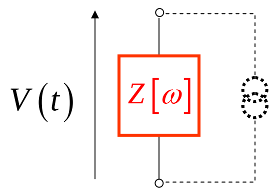
CLASSICAL RESPONSE FUNCTION



$$S_{II}[\omega] = \hbar\omega\left[\coth\left(\frac{\hbar\omega}{2k_B T}\right)+1\right]\mathrm{Re}\left(Y[\omega]\right)$$

08-V-22

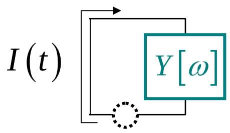
GENERALIZATION OF QUANTUM FLUCTUATION-DISSIPATION THEOREM TO ARBITRARY IMPEDANCE OR ADMITTANCE



$$S_{VV}[\omega] = \hbar\omega\left[\coth\left(\frac{\hbar\omega}{2k_BT}\right)+1\right]\mathrm{Re}\left(Z[\omega]\right)$$

QUANTUM BLACK-BODY BOSE FACTOR

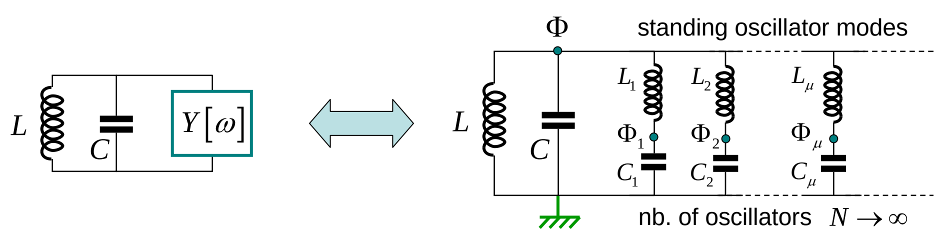
CLASSICAL RESPONSE FUNCTION



$$S_{II}[\omega] = \hbar\omega\left[\coth\left(\frac{\hbar\omega}{2k_BT}\right)+1\right]\mathrm{Re}\left(Y[\omega]\right)$$

08-V-22a

FLUCTUATIONS OF OSCILLATOR FOR AN ARBITRARY ADMITTANCE



$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} + \sum_{\mu=1}^{N}\left[\frac{\hat{Q}_\mu}{2C_\mu} + \frac{\left(\hat{\Phi}_\mu - \hat{\Phi}\right)^2}{2L_\mu}\right]$$

$$\mathrm{Re}\left[Y\left[\omega_\mu\right]\right] = \sqrt{\frac{C_\mu}{L_\mu}} \qquad \omega_\mu = \sqrt{\frac{1}{L_\mu C_\mu}}$$

$$Z_{tot}[\omega] = \frac{1}{\dfrac{1}{jL\omega} + jC\omega + Y[\omega]} \qquad (j = -i)$$

$$\left\langle \hat{\Phi}^2 \right\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \frac{\hbar Z_T[\omega]}{2\omega}\coth\left(\frac{\hbar\omega}{2k_BT}\right)$$

$$\left\langle \hat{Q}^2 \right\rangle = \frac{C^2}{2\pi}\int_{-\infty}^{+\infty} d\omega \frac{\hbar\omega Z_T[\omega]}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)$$

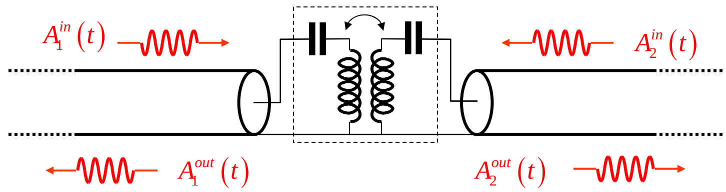
08-V-23c

OUTLINE

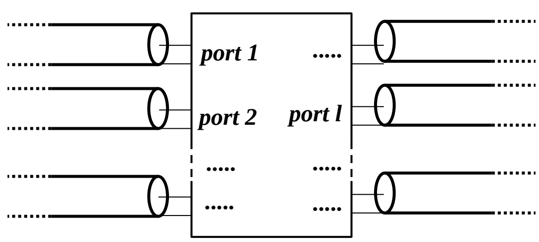
1. Where we stand so far, purpose of this lecture
2. Fluctuations of damped harmonic oscillator
3. Hamiltonian approach of Caldeira and Leggett
4. Properties of multiple ports scattering matrix
5. Non-linear active circuits

MULTIPLE PORT CIRCUITS

Example:



In general:



SCATTERING MATRIX FOR PASSIVE LINEAR CIRCUIT

same frequency

$$\begin{bmatrix} A_1^{out}[\omega] \\ A_2^{out}[\omega] \\ \cdots \\ A_l^{out}[\omega] \\ \cdots \end{bmatrix} = \begin{bmatrix} s_{11}[\omega] & s_{12}[\omega] & \cdots & s_{1l}[\omega] & \cdots \\ s_{21}[\omega] & s_{22}[\omega] & \cdots & s_{2l}[\omega] & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{l1}[\omega] & s_{l2}[\omega] & \cdots & s_{ll}[\omega] & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} A_1^{in}[\omega] \\ A_2^{in}[\omega] \\ \cdots \\ A_l^{in}[\omega] \\ \cdots \end{bmatrix}$$

OUTGOING AMPLITUDES — "S" PARAMETERS, SCATTERING MATRIX — INCOMING AMPLITUDES

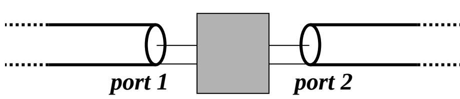
CAUSALITY ⇨ POLES OF S MATRIX IN LOWER-HALF PLANE

INFORMATION CONSERVATION + ENERGY CONSERVATION ⇨ UNITARITY OF S MATRIX

$$S^{\dagger}S=1$$

08-V-25a

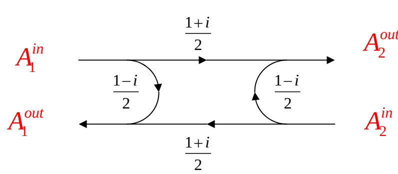
SCATTERING MATRIX FOR NON-DISSIPATIVE LINEAR 2-PORT



$$S = \begin{bmatrix} e^{i\alpha}|r| & e^{i\beta}\sqrt{1-r^2} \\ e^{i\gamma}\sqrt{1-r^2} & e^{i\delta}|r| \end{bmatrix}$$

Unitarity constraint: $\alpha + \delta - \beta - \gamma = \pi \bmod 2\pi$

Fully balanced 2-port:



~~$\begin{bmatrix} \cos\vartheta & i\sin\vartheta \\ -i\sin\vartheta & \cos\vartheta \end{bmatrix}$~~

$\begin{bmatrix} \cos\vartheta & i\sin\vartheta \\ i\sin\vartheta & \cos\vartheta \end{bmatrix}$ OK!

Note that S_{11} and S_{22} are reflection coefficients for port 1 and 2 provided the other port is terminated by characteristic impedance

08-V-26

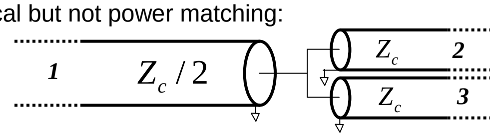
SYMMETRIES OF S MATRIX: CONSTRAINTS FOR LINEAR 3-PORT (NO PERFECT DIVIDER EXISTS)

THEOREM: IT IS IMPOSSIBLE FOR A NON-DISSIPATIVE 3-PORT TO BE SIMULTANEOUSLY POWER-MATCHING AND RECIPROCAL.

POWER- MATCHING: $\forall l, s_{ll} = 0$ RECIPROCITY: ${}^{t}\mathrm{S} = \mathrm{S}$

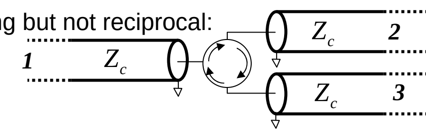
- Non-dissipative and reciprocal but not power matching:

$s_{22} = s_{33} \neq 0$



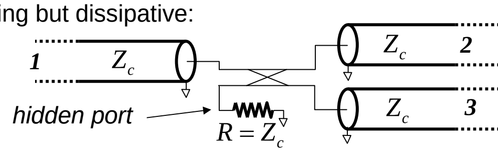
- Non-dissipative and power-matching but not reciprocal:

Circulator:



- Reciprocal and power-matching but dissipative:

Wilkinson divider:



08-V-27c

OUTLINE

1. Where we stand so far, purpose of this lecture
2. Fluctuations of damped harmonic oscillator
3. Hamiltonian approach of Caldeira and Leggett
4. Properties of multiple ports scattering matrix
5. Non-linear active circuits

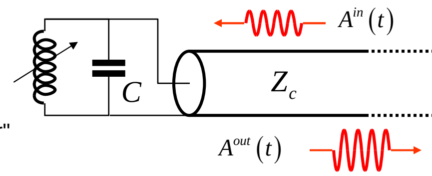
08-V-5e

ACTIVE LINEAR 1-PORT

B. Yurke, in " Quantum Squeezing" , Springer (2004)

Simplest example:

$$L(t) = L + \delta L \sin(2\omega_0 t)$$



$$\omega_0 = \frac{1}{\sqrt{LC}}$$

"Degenerate Parametric Amplifier"

$$\Gamma = \frac{1}{2Z_c C}$$

$$\ddot{\Phi} + 2\Gamma\dot{\Phi} + \omega_0^2\Phi + i\omega_0^2\chi\Phi\left(e^{2i\omega_0 t} - e^{-2i\omega_0 t}\right) = 4\Gamma V^{in}(t) \qquad V^{out}(t) = \dot{\Phi} - V^{in}(t)$$

Harmonic balance $\Rightarrow$

$$2i\omega_0\Gamma\Phi[\omega_0] - i\omega_0^2\chi\Phi[-\omega_0] = 4\Gamma V^{in}[\omega_0]$$

After a few steps:

$$\begin{cases} A^{out}[\omega_0] = \dfrac{1+\zeta^2}{1-\zeta^2}A^{in}[\omega_0] + \dfrac{2\zeta}{1-\zeta^2}A^{in}[-\omega_0] \\ A^{out}[-\omega_0] = \dfrac{1+\zeta^2}{1-\zeta^2}A^{in}[-\omega_0] + \dfrac{2\zeta}{1-\zeta^2}A^{in}[+\omega_0] \end{cases}$$

$$\zeta = \frac{1}{4}\frac{\delta L}{L}\frac{\omega_0}{\Gamma} < 1$$

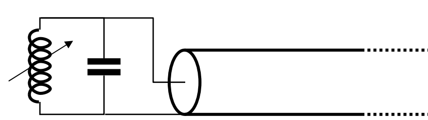
Generalized scattering matrix

$$\begin{bmatrix} A^{out}[\omega_0] \\ A^{out}[-\omega_0] \end{bmatrix} = \begin{bmatrix} c & s \\ s & c \end{bmatrix} \begin{bmatrix} A^{in}[\omega_0] \\ A^{in}[-\omega_0] \end{bmatrix} \qquad c^2 - s^2 = 1$$

08-V-28c

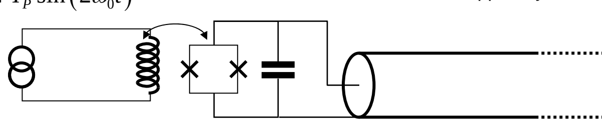
NON-LINEAR CIRCUIT CAN BEHAVE LINEARLY FOR WEAK SIGNALS: SQUID

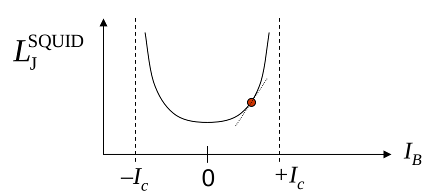
$$L(t) = L + \delta L \sin(2\omega_0 t)$$



NEC group: Yamamoto et al., Appl. Phys. Lett. 93, 042510 (2008)

$$I_B(t) = I_B + I_P \sin(2\omega_0 t)$$





IN THE VICINITY OF OPERATING POINT SQUID IS A VARIABLE INDUCTOR

08-V-29

SCATTERING MATRIX FOR ACTIVE LINEAR 2-PORTS

different frequencies

$$\begin{bmatrix} a_1^{out}[\omega_1] \\ a_1^{out}[-\omega_1] \\ a_2^{out}[\omega_2] \\ a_2^{out}[-\omega_2] \end{bmatrix} = \begin{bmatrix} r_1 & s_1 & t_{12} & u_{12} \\ s_1^* & r_1^* & u_{12}^* & t_{12}^* \\ t_{21} & u_{21} & r_2 & s_2 \\ u_{21}^* & t_{21}^* & s_2^* & r_2^* \end{bmatrix} \begin{bmatrix} a_1^{in}[\omega_1] \\ a_1^{in}[-\omega_1] \\ a_2^{in}[\omega_2] \\ a_2^{in}[-\omega_2] \end{bmatrix}$$

$$a[\omega] \sim \frac{1}{\sqrt{\omega}} A[\omega]$$

INFORMATION CONSERVATION ⇨ S MATRIX IS SYMPLECTIC $\quad {}^tSJS = J$

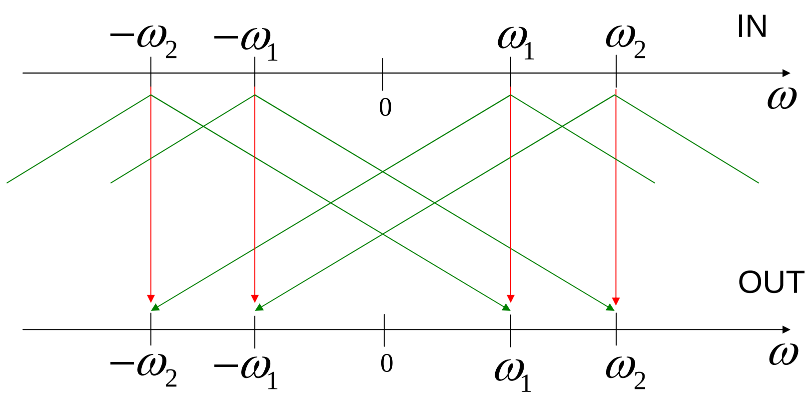
(ENERGY IS NOT CONSERVED, S IS NOT UNITARY)

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

central to antisymmetric bilinear forms such as Poisson Brackets (and commutators)

08-V-31

IN NON-LINEAR ACTIVE DEVICES, SIGNALS CAN SCATTER BETWEEN DIFFERENT FREQUENCIES



Pump frequency: Ω_p

$$\omega_1 + \omega_2 = 0 \bmod \Omega_p$$

08-V-30

TOPICS OF NEXT LECTURE:

1) AMPLIFICATION OF QUANTUM SIGNALS:
ULTIMATE SENSITIVITY

2) SQUEEZING OF QUANTUM NOISE

08-V-32

END OF LECTURE